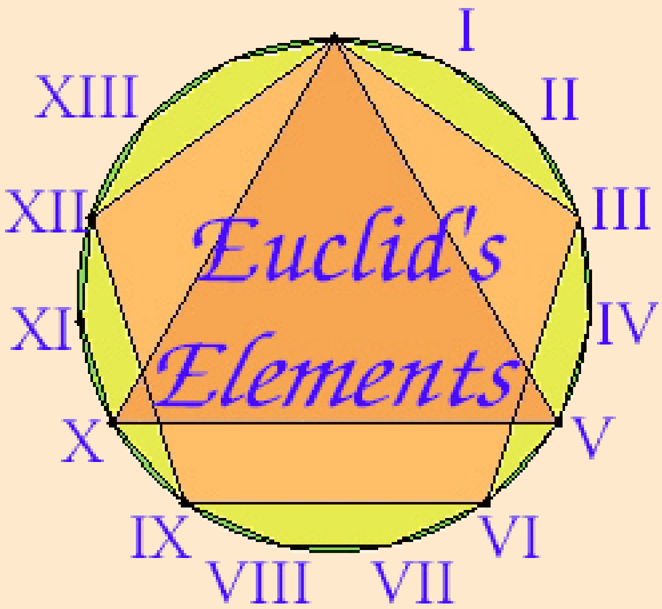

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Prematter

Euclid's Elements

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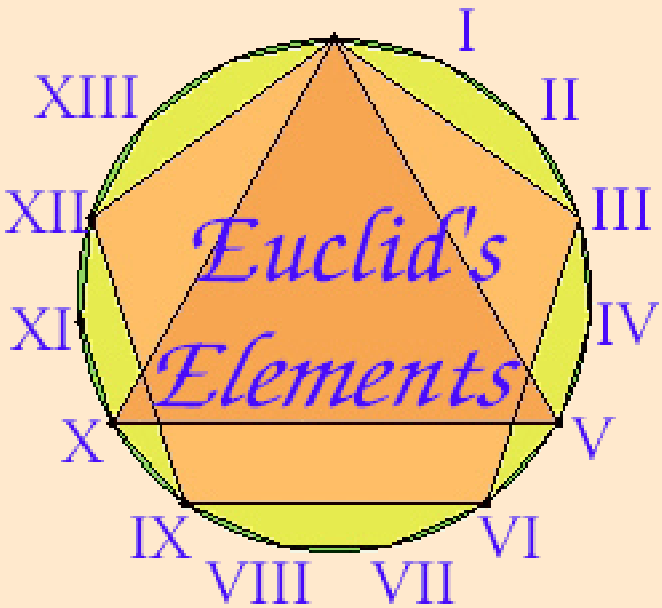
These pages are located at http://aleph0.clarku.edu/~djoyce/ java/elements/elements.html.



Book I

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Definitions

Definition 1.

A *point* is that which has no part.

Definition 2.

A *line* is breadthless length.

Definition 3.

The ends of a line are points.

Definition 4.

A *straight line* is a line which lies evenly with the points on itself.

Definition 5.

A *surface* is that which has length and breadth only.

Definition 6.

The edges of a surface are lines.

Definition 7.

A *plane surface* is a surface which lies evenly with the straight lines on itself.

Definition 8.

A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Definition 9.

And when the lines containing the angle are straight, the angle is called *rectilinear.*

Definition 10.

When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right,* and the straight line standing on the other is called a *perpendicular* to that on which it stands.

Definition 11.

An *obtuse angle* is an angle greater than a right angle.

Definition 12.

An *acute angle* is an angle less than a right angle.

Definition 13.

A *boundary* is that which is an extremity of anything.

Definition 14.

A *figure* is that which is contained by any boundary or boundaries.

Definition 15.

A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Definition 16.

And the point is called the *center* of the circle.

Definition 17.

A *diameter* of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

Definition 18.

A *semicircle* is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

Definition 19.

Rectilinear figures are those which are contained by straight lines, *trilateral* figures being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines.

Definition 20.

Of trilateral figures, an *equilateral triangle* is that which has its three sides equal, an *isosceles triangle* that which has two of its sides alone equal, and a *scalene triangle* that which has its three sides unequal.

Definition 21.

Further, of trilateral figures, a *right-angled triangle* is that which has a right angle, an *obtuse-angled triangle* that which has an obtuse angle, and an *acute-angled triangle* that which has its three angles acute.

Definition 22.

Of quadrilateral figures, a *square* is that which is both equilateral and right-angled; an *oblong* that which is right-angled but not equilateral; a *rhombus* that which is equilateral but not right-angled; and a *rhomboid* that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called *trapezia.*

Definition 23

Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Postulates

Let the following be postulated:

Postulate 1.

To draw a straight line from any point to any point.

Postulate 2.

To produce a finite straight line continuously in a straight line.

Postulate 3.

To describe a circle with any center and radius.

Postulate 4.

That all right angles equal one another.

Postulate 5.

That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Common Notions

Common notion 1.

Things which equal the same thing also equal one another.

Common notion 2.

If equals are added to equals, then the wholes are equal.

Common notion 3.

If equals are subtracted from equals, then the remainders are equal.

Common notion 4.

Things which coincide with one another equal one another.

Common notion 5.

The whole is greater than the part.

Propositions

Proposition 1.

To construct an equilateral triangle on a given finite straight line.

Proposition 2.

To place a straight line equal to a given straight line with one end at a given point.

Proposition 3.

To cut off from the greater of two given unequal straight lines a straight line equal to the less.

Proposition 4.

If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

Proposition 5.

In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

Proposition 6.

If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

Proposition 7.

Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.

Proposition 8.

If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.

Proposition 9.

To bisect a given rectilinear angle.

Proposition 10.

To bisect a given finite straight line.

Proposition 11.

To draw a straight line at right angles to a given straight line from a given point on it.

Proposition 12.

To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

Proposition 13.

If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

Proposition 14.

If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

Proposition 15.

If two straight lines cut one another, then they make the vertical angles equal to one another.

Corollary. If two straight lines cut one another, then they will make the angles at the point of section equal to four right angles.

Proposition 16.

In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

Proposition 17.

In any triangle the sum of any two angles is less than two right angles.

Proposition 18.

In any triangle the angle opposite the greater side is greater.

Proposition 19.

In any triangle the side opposite the greater angle is greater.

Proposition 20.

In any triangle the sum of any two sides is greater than the remaining one.

Proposition 21.

If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

Proposition 22.

To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one.

Proposition 23.

To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.

Proposition 24.

If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.

Proposition 25.

If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have the one of the angles contained by the equal straight lines greater than the other.

Proposition 26.

If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.

Proposition 27.

If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

Proposition 28.

If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

Proposition 29.

A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

Proposition 30.

Straight lines parallel to the same straight line are also parallel to one another.

Proposition 31.

To draw a straight line through a given point parallel to a given straight line.

Proposition 32.

In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.

Proposition 33.

Straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.

Proposition 34.

In parallelogrammic areas the opposite sides and angles equal one another, and the diameter bisects the areas.

Proposition 35.

Parallelograms which are on the same base and in the same parallels equal one another.

Proposition 36.

Parallelograms which are on equal bases and in the same parallels equal one another.

Proposition 37.

Triangles which are on the same base and in the same parallels equal one another.

Proposition 38.

Triangles which are on equal bases and in the same parallels equal one another.

Proposition 39.

Equal triangles which are on the same base and on the same side are also in the same parallels.

Proposition 40.

Equal triangles which are on equal bases and on the same side are also in the same parallels.

Proposition 41.

If a parallelogram has the same base with a triangle and is in the same parallels, then the parallelogram is double the triangle.

Proposition 42.

To construct a parallelogram equal to a given triangle in a given rectilinear angle.

Proposition 43.

In any parallelogram the complements of the parallelograms about the diameter equal one another.

Proposition 44.

To a given straight line in a given rectilinear angle, to apply a parallelogram equal to a given triangle.

Proposition 45.

To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.

Proposition 46.

To describe a square on a given straight line.

Proposition 47.

In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Proposition 48.

> If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.

Guide

About the Definitions

The *Elements* begins with a list of definitions. Some of these indicate little more than certain concepts will be discussed, such as Def.I.1, Def.I.2, and Def.I.5, which introduce the terms point, line, and surface. (Note that for Euclid, the concept of line includes curved lines.) Others are substantial definitions which actually describe new concepts in terms of old ones. For example, Def.I.10 defines a *right angle* as one of two equal adjacent angles made when one straight line meets another. Other definitions look like they're substantial, but actually are not. For instance, Def.I.4 says a *straight line* "is a line which lies evenly with the points on itself." No where in the *Elements* is the defining phrase "which lies evenly with the points on itself" applicable. Thus, this definition indicates, at most, that some lines under discussion will be straight lines.

It has been suggested that the definitions were added to the *Elements* sometime after Euclid wrote them. Another possibility is that they are actually from a different work, perhaps older. In Def.I.22 special kinds of quadrilaterals are defined including square, oblong (a rectangle that are not squares), rhombus (equilateral but not a square), and rhomboid (parallelogram but not a rhombus). Except for squares, these other shapes are not mentioned in the *Elements.* Euclid does use parallelograms, but they're not defined in this definition. Also, the exclusive nature of some of these terms—the part that indicates not a square—is contrary to Euclid's practice of accepting squares and rectangles as kinds of parallelograms.

About the Postulates

Following the list of definitions is a list of postulates. Each postulate is an axiom—which means a statement which is accepted without proof— specific to the subject matter, in this case, plane geometry. Most of them are constructions. For instance, Post.I.1 says a straight line can be drawn between two points, and Post.I.3 says a circle can be drawn given a specified point to be the center and another point to be on the circumference. The fourth postulate, Post.I.4, is not a constuction, but says that all right angles are equal.

About magnitudes and the Common Notions

The Common Notions are also axioms, but they refer to magnitudes of various kinds. The kind of magnitude that appears most frequently is that of straight line. Other important kinds are rectilinear angles and areas (plane figures). Later books include other kinds.

In proposition III.16 (but nowhere else) angles with curved sides are compared with rectilinear angles which shows that rectilinear angles are to be considered as a special kind of plane angle. That agrees with Euclid's definition of them in I.Def.9 and I.Def.8.

Also in Book III, parts of circumferences of circles, that is, arcs, appear as magnitudes. Only arcs of equal circles can be compared or added, so arcs of equal circles comprise a kind of magnitude, while arcs of unequal circles are magnitudes of different kinds. These kinds are all different from straight lines. Whereas areas of figures are comparable, different kinds of curves are not.

Book V includes the general theory of ratios. No particular kind of magnitude is specified in that book. It may come as a surprise that ratios do not themselves form a kind of magnitude since they can be compared, but they cannot be added. See the guide on Book V for more information.

Number theory is treated in Books VII through IX. It could be considered that numbers form a kind of magnitude as pointed out by Aristotle.

Beginning in Book XI, solids are considered, and they form the last kind of magnitude discussed in the *Elements.*

The propositions

Following the definitions, postulates, and common notions, there are 48 propositions. Each of these propositions includes a statement followed by a proof of the statement. Each statement of the proof is logically justified by a definition, postulate, common notion, or an earlier proposition that has already been proven. There are gaps in the logic of some of the proofs, and these are mentioned in the commenaries after the propositions. Also included in the proof is a diagram illustrating the proof.

Some of the propositions are constructions. A construction depends, ultimately, on the constructive postulates about drawing lines and circles. The first part of a proof for a constuctive proposition is how to perform the construction. The rest of the proof (usually the longer part), shows that the proposed construction actually satisfies the goal of the proposition. In the list of propositions in each book, the constructions are displayed in red.

Most of the propositions, however, are not constructions. Their statements say that under certain conditions, certain other conditions logically follow. For example, Prop.I.5 says that if a triangle has the property that two of its sides are equal, then it follows that the angles opposite these sides (called the "base angles") are also equal. Even the propositions that are not constructions may have constructions included in their proofs since auxillary lines or circles may be needed in the explanation. But the bulk of the proof is, as for the constructive propositions, a sequence of statements that are logically justified and which culminates in the statement of the proposition.

Logical structure of Book I

The various postulates and common notions are frequently used in Book I. Only two of the propositions rely solely on the postulates and axioms, namely, I.1 and I.4. The logical chains of propositions in Book I are longer than in the other books; there are long sequences of propositions each relying on the previous.

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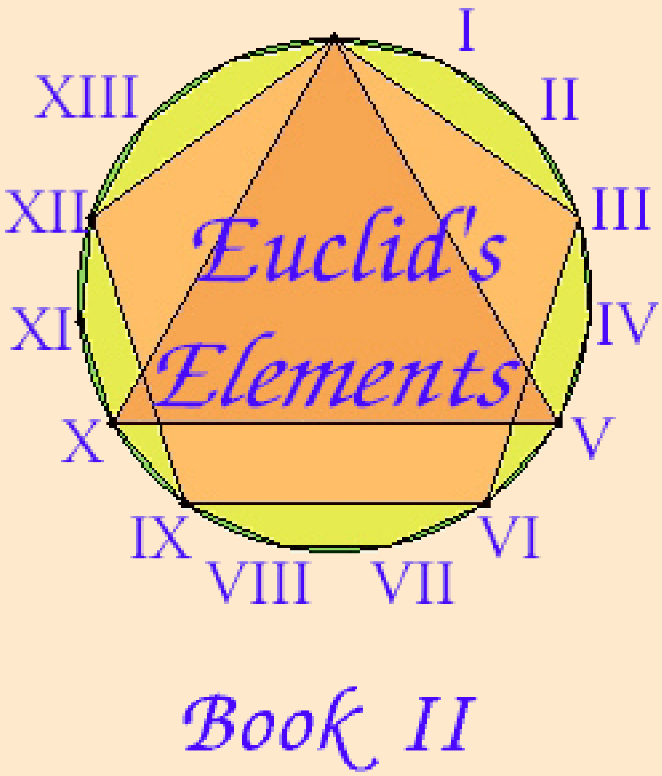

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Definitions

Definition 1.

Any rectangular parallelogram is said to be *contained* by the two straight lines containing the right angle.

Definition 2

And in any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called a *gnomon.*

Propositions

Proposition 1.

If there are two straight lines, and one of them is cut into any number of segments whatever, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by the uncut straight line and each of the segments.

Proposition 2.

If a straight line is cut at random, then the sum of the rectangles contained by the whole and each of the segments equals the square on the whole.

Proposition 3.

If a straight line is cut at random, then the rectangle contained by the whole and one of the segments equals the sum of the rectangle contained by the segments and the square on the aforesaid segment.

Proposition 4.

If a straight line is cut at random, the square on the whole equals the squares on the segments plus twice the rectangle contained by the segments.

Proposition 5.

If a straight line is cut into equal and unequal segments, then the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section equals the square on the half.

Proposition 6.

If a straight line is bisected and a straight line is added to it in a straight line, then the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half equals the square on the straight line made up of the half and the added straight line.

Proposition 7.

If a straight line is cut at random, then the sum of the square on the whole and that on one of the segments equals twice the rectangle contained by the whole and the said segment plus the square on the remaining segment.

Proposition 8.

If a straight line is cut at random, then four times the rectangle contained by the whole and one of the segments plus the square on the remaining segment equals the square described on the whole and the aforesaid segment as on one straight line.

Proposition 9.

If a straight line is cut into equal and unequal segments, then the sum of the squares on the unequal segments of the whole is double the sum of the square on the half and the square on the straight line between the points of section.

Proposition 10.

If a straight line is bisected, and a straight line is added to it in a straight line, then the square on the whole with the added straight line and the square on the added straight line both together are double the sum of the square on the half and the square described on the straight line made up of the half and the added straight line as on one straight line.

Proposition 11.

To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment.

Proposition 12.

In obtuse-angled triangles the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.

Proposition 13.

In acute-angled triangles the square on the side opposite the acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acutc angle.

Proposition 14.

To construct a square equal to a given rectilinear figure.

Guide to Book II

The subject matter of Book II is usually called "geometric algebra." The first ten propositions of Book II can be easily interpreted in modern algebraic notation. Of course, in doing so the geometric flavor of the propositions is lost. Nonetheless, restating them algebraically can aid in understanding them. The equations are all quadratic equations since the geometry is plane geometry.

II.1. If $y = y_1 + y_2 + ... + y_n$, then $xy = xy_1 + xy_2 + ... + xy_n$. This can be stated in a single identity as

$$x (y_1 + y_2 + ... + y_n) = xy_1 + xy_2 + ... + xy_n.$$

II.2. If $x = y + z$, then $x^2 = xy + xz$. This can be stated in various ways in an identity of two variables. For instance,

$$(y + z)^2 = (y + z) y + (y + z) z,$$

or

$$x^2 = xy + x (x - y).$$

II.3. If $x = y + z$, then $xy = yz + y^2$. Equivalent identities are

$$(y + z)y = yz + y^2,$$

and

$$xy = y (x - y) + y^2.$$

II.4. If $x = y + z$, then $x^2 = y^2 + z^2 + 2yz$. As an identity,

$$(y + z)^2 = y^2 + z^2 + 2yz.$$

II.5 and II.6. $(y + z) (y - z) + z^2 = y^2$.

II.7. if $x = y + z$, then $x^2 + z^2 = 2xz + y^2$. As an identity,

$$x^2 + z^2 = 2xz + (x - z)^2.$$

II.8. If $x = y + z$, then $4xy + z^2 = (x + y)^2$. As an identity,

$$4xy + (x - y)^2 = (x + y)^2.$$

II.9 and II.10. $(y + z)^2 + (y - z)^2 = 2 (y^2 + z^2)$.

The remaining four propositions are of a slightly different nature. Proposition II.11 cuts a line into two parts which solves the equation $a (a - x) = x^2$ geometrically. Propositions II.12 and II.13 are recognizable as geometric forms of the law of cosines which is a generalization of I.47. The last propostion II.14 constructs a square equal to a given rectilinear figure thereby completeing the theory of areas begun in Book I.

Logical structure of Book II

The proofs of the propositions in Book II heavily rely on the propositions in Book I involving right angles and parallel lines, but few others. For instance, the important congruence theorems for triangles, namely I.4, I.8, and I.26, are not invoked even once. This is understandable considering Book II is mostly algebra interpreted in the theory of geometry.

The first ten propositions in Book II were written to be logically independent, but they could have easily been written in logical chains which, perhaps, would have shortened the exposition a little. The remaining four propositions each depend on one of the first ten.

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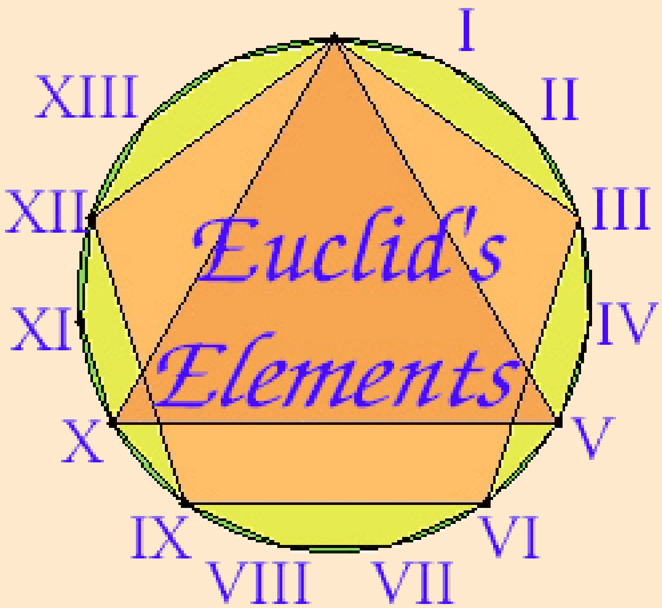

Book III

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Definitions

Definition 1.

Equal circles are those whose diameters are equal, or whose radii are equal.

Definition 2.

A straight line is said to *touch* a circle which, meeting the circle and being produced, does not cut the circle.

Definition 3.

Circles are said to *touch* one another which meet one another but do not cut one another.

Definition 4.

Straight lines in a circle are said to be *equally distant* from the center when the perpendiculars drawn to them from the center are equal.

Definition 5.

And that straight line is said to be at a *greater distance* on which the greater perpendicular falls.

Definition 6.

A *segment* of a circle is the figure contained by a straight line and a circumference of a circle.

Definition 7.

An *angle of a segment* is that contained by a straight line and a circumference of a circle.

Definition 8.

An *angle in a segment* is the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the ends of the straight line which is the base of the segment, is contained by the straight lines so joined.

Definition 9.

And, when the straight lines containing the angle cut off a circumference, the angle is said to *stand upon* that circumference.

Definition 10.

A *sector* of a circle is the figure which, when an angle is constructed at the center of the circle, is contained by the straight lines containing the angle and the circumference cut off by them.

Definition 11.

Similar segments of circles are those which admit equal angles, or in which the angles equal one another.

Propositions

Proposition 1.

To find the center of a given circle.

Corollary. If in a circle a straight line cuts a straight line into two equal parts and at right angles, then the center of the circle lies on the cutting straight line.

Proposition 2.

If two points are taken at random on the circumference of a circle, then the straight line joining the points falls within the circle.

Proposition 3.

If a straight line passing through the center of a circle bisects a straight line not passing through the center, then it also cuts it at right angles; and if it cuts it at right angles, then it also bisects it.

Proposition 4.

If in a circle two straight lines which do not pass through the center cut one another, then they do not bisect one another.

Proposition 5.

If two circles cut one another, then they do not have the same center.

Proposition 6.

If two circles touch one another, then they do not have the same center.

Proposition 7.

If on the diameter of a circle a point is taken which is not the center of the circle, and from the point straight lines fall upon the circle, then that is greatest on which passes through the center, the remainder of the same diameter is least, and of the rest the nearer to the straight line through the center is always greater than the more remote; and only two equal straight lines fall from the point on the circle, one on each side of the least straight line.

Proposition 8.

If a point is taken outside a circle and from the point straight lines are drawn through to the circle, one of which is through the center and the others are drawn at random, then, of the straight lines which fall on the concave circumference, that through the center is greatest, while of the rest the nearer to that through the center is always greater than the more remote, but, of the straight lines falling on the convex circumference, that between the point and the diameter is least, while of the rest the nearer to the least is always less than the more remote; and only two equal straight lines fall on the circle from the point, one on each side of the least.

Proposition 9.

If a point is taken within a circle, and more than two equal straight lines fall from the point on the circle, then the point taken is the center of the circle.

Proposition 10.

A circle does not cut a circle at more than two points.

Proposition 11.

If two circles touch one another internally, and their centers are taken, then the straight line joining their centers, being produced, falls on the point of contact of the circles.

Proposition 12.

If two circles touch one another externally, then the straight line joining their centers passes through the point of contact.

Proposition 13.

A circle does not touch another circle at more than one point whether it touches it internally or externally..

Proposition 14.

Equal straight lines in a circle are equally distant from the center, and those which are equally distant from the center equal one another.

Proposition 15.

Of straight lines in a circle the diameter is greatest, and of the rest the nearer to the center is always greater than the more remote.

Proposition 16.

The straight line drawn at right angles to the diameter of a circle from its end will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed, further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle.

Corollary. From this it is manifest that the straight line drawn at right angles to the diameter of a circle from its end touches the circle.

Proposition 17.

From a given point to draw a straight line touching a given circle.

Proposition 18.

If a straight line touches a circle, and a straight line is joined from the center to the point of contact, the straight line so joined will be perpendicular to the tangent.

Proposition 19.

If a straight line touches a circle, and from the point of contact a straight line is drawn at right angles to the tangent, the center of the circle will be on the straight line so drawn.

Proposition 20.

In a circle the angle at the center is double the angle at the circumference when the angles have the same circumference as base.

Proposition 21.

In a circle the angles in the same segment equal one another.

Proposition 22.

The sum of the opposite angles of quadrilaterals in circles equals two right angles.

Proposition 23.

On the same straight line there cannot be constructed two similar and unequal segments of circles on the same side.

Proposition 24.

Similar segments of circles on equal straight lines equal one another.

Proposition 25.

Given a segment of a circle, to describe the complete circle of which it is a segment.

Proposition 26.

In equal circles equal angles stand on equal circumferences whether they stand at the centers or at the circumferences.

Proposition 27.

In equal circles angles standing on equal circumferences equal one another whether they stand at the centers or at the circumferences.

Proposition 28.

In equal circles equal straight lines cut off equal circumferences, the greater circumference equals the greater and the less equals the less.

Proposition 29.

In equal circles straight lines that cut off equal circumferences are equal.

Proposition 30.

To bisect a given circumference.

Proposition 31.

In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; further the angle of the greater segment is greater than a right angle, and the angle of the less segment is less than a right angle.

Proposition 32.

If a straight line touches a circle, and from the point of contact there is drawn across, in the circle, a straight line cutting the circle, then the angles which it makes with the tangent equal the angles in the alternate segments of the circle.

Proposition 33.

On a given straight line to describe a segment of a circle admitting an angle equal to a given rectilinear angle.

Proposition 34.

From a given circle to cut off a segment admitting an angle equal to a given rectilinear angle.

Proposition 35.

If in a circle two straight lines cut one another, then the rectangle contained by the segments of the one equals the rectangle contained by the segments of the other.

Proposition 36.

If a point is taken outside a circle and two straight lines fall from it on the circle, and if one of them cuts the circle and the other touches it, then the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the tangent.

Proposition 37.

If a point is taken outside a circle and from the point there fall on the circle two straight lines, if one of them cuts the circle, and the other falls on it, and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the straight line which falls on the circle, then the straight line which falls on it touches the circle.

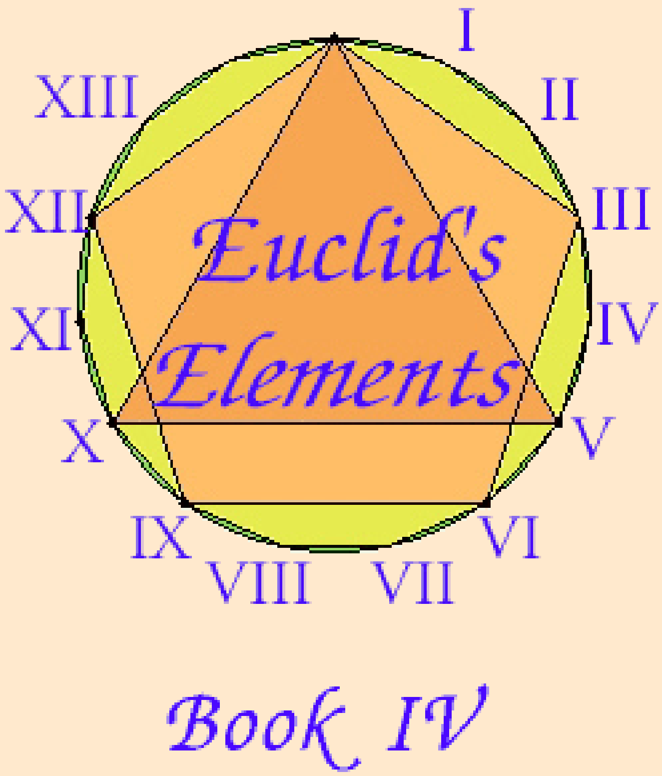
Next book: Book IV

Previous: Book II

Select from Book III

Select book

Select topic



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Definitions

Definition 1.

A rectilinear figure is said to be *inscribed in a rectilinear figure* when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed.

Definition 2.

Similarly a figure is said to be *circumscribed about a figure* when the respective sides of the circumscribed figure pass through the respective angles of that about which it is circumscribed.

Definition 3.

A rectilinear figure is said to be *inscribed in a circle* when each angle of the inscribed figure lies on the circumference of the circle.

Definition 4.

A rectilinear figure is said to be *circumscribed about a circle* when each side of the circumscribed figure touches the circumference of the circle.

Definition 5.

Similarly a circle is said to be *inscribed in a figure* when the circumference of the circle touches each side of the figure in which it is inscribed.

Definition 6.

A circle is said to be *circumscribed about a figure* when the circumference of the circle passes through each angle of the figure about which it is circumscribed.

Definition 7.

A straight line is said to be *fitted into a circle* when its ends are on the circumference of the circle.

Propositions

Proposition 1.

To fit into a given circle a straight line equal to a given straight line which is not greater than the diameter of the circle.

Proposition 2.

To inscribe in a given circle a triangle equiangular with a given triangle.

Proposition 3.

To circumscribe about a given circle a triangle equiangular with a given triangle.

Proposition 4.

To inscribe a circle in a given triangle.

Proposition 5.

To circumscribe a circle about a given triangle.

Corollary. When the center of the circle falls within the triangle, the triangle is acute-angled; when the center falls on a side, the triangle is right-angled; and when the center of the circle falls outside the triangle, the triangle is obtuse-angled.

Proposition 6.

To inscribe a square in a given circle.

Proposition 7.

To circumscribe a square about a given circle.

Proposition 8.

To inscribe a circle in a given square.

Proposition 9.

To circumscribe a circle about a given square.

Proposition 10.

To construct an isosceles triangle having each of the angles at the base double the remaining one.

Proposition 11.

To inscribe an equilateral and equiangular pentagon in a given circle.

Proposition 12.

To circumscribe an equilateral and equiangular pentagon about a given circle.

Proposition 13.

To inscribe a circle in a given equilateral and equiangular pentagon.

Proposition 14.

To circumscribe a circle about a given equilateral and equiangular pentagon.

Proposition 15.

To inscribe an equilateral and equiangular hexagon in a given circle.

Corollary. The side of the hexagon equals the radius of the circle.

And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle an equilateral and equiangular hexagon in conformity with what was explained in the case of the pentagon.

And further by means similar to those explained in the case of the pentagon we can both inscribe a circle in a given hexagon and circumscribe one about it.

Proposition 16.

To inscribe an equilateral and equiangular fifteen-angled figure in a given circle.

Corollary. And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle a fifteen-angled figure which is equilateral and equiangular.

And further, by proofs similar to those in the case of the pentagon, we can both inscribe a circle in the given fifteen-angled figure and circumscribe one about it.

Guide to Book IV

All but two of the propositions in this book are constructions to inscribe or circumscribe figures.

Figure	Inscribe figure in circle	Circumscribe figure about circle	Inscribe circle in figure	Circumscribe circle about figure
Triangle	IV.2	IV.3	IV.4	IV.5
Square	IV.6	IV.7	IV.8	IV.9
Regular pentagon	IV.11	IV.12	IV.13	IV.14
Regular hexagon	IV.15	IV.15,Cor	IV.15,Cor	IV.15,Cor
Regular 15-gon	IV.16	IV.16,Cor	IV.16,Cor	IV.16,Cor

There are only two other propositions. Proposition IV.1 is a basic construction to fit a line in a circle, and proposition IV.10 constructs a particular triangle needed in the construction of a regular pentagon.

Logical structure of Book IV

The proofs of the propositions in Book IV rely heavily on the propositions in Books I and III. Only one proposition from Book II is used and that is the construction in II.11 used in proposition IV.10 to construct a particular triangle needed in the construction of a regular pentagon.

Most of the propositions of Book IV are logically independent of each other. There is a short chain of deductions, however, involving the construction of regular pentagons.

Dependencies within Book IV	
1, 5	10
2, 10	11
11	12
1, 2, 11	16

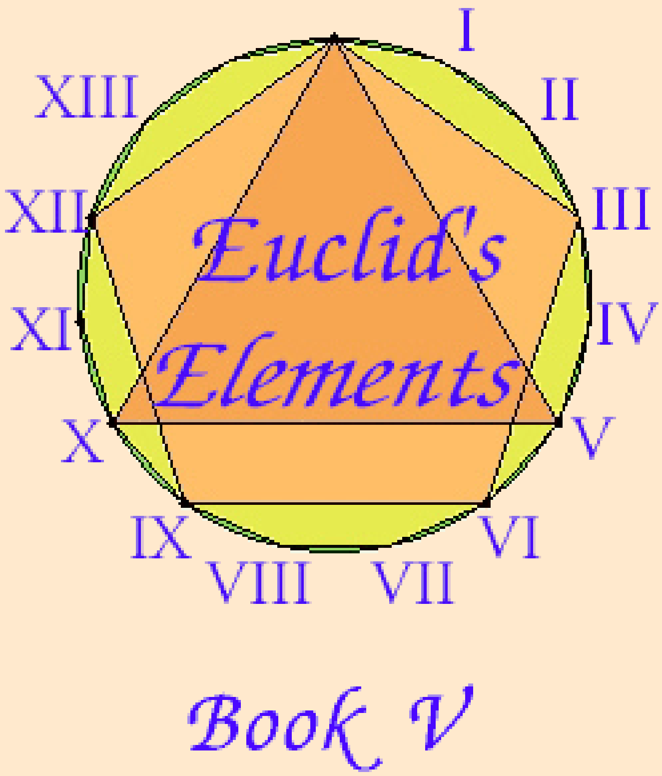
Next book: Book V

Previous: Book III

Select from Book IV

Select book

Select topic



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Definitions

Definition 1

A magnitude is a *part* of a magnitude, the less of the greater, when it measures the greater.

Definition 2

The greater is a *multiple* of the less when it is measured by the less.

Definition 3

A *ratio* is a sort of relation in respect of size between two magnitudes of the same kind.

Definition 4

Magnitudes are said to *have a ratio* to one another which can, when multiplied, exceed one another.

Definition 5

Magnitudes are said to be *in the same ratio,* the first to the second and the third to the fourth, when, if any equimultiples whatever are taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

Definition 6

Let magnitudes which have the same ratio be called *proportional.*

Definition 7

When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a *greater ratio* to the second than the third has to the fourth.

Definition 8

A proportion in three terms is the least possible.

Definition 9

When three magnitudes are proportional, the first is said to have to the third the *duplicate ratio* of that which it has to the second.

Definition 10

When four magnitudes are continuously proportional, the first is said to have to the fourth the *triplicate ratio* of that which it has to the second, and so on continually, whatever be the proportion.

Definition 11

Antecedents are said to *correspond* to antecedents, and consequents to consequents.

Definition 12

Alternate ratio means taking the antecedent in relation to the antecedent and the consequent in relation to the consequent.

Definition 13

Inverse ratio means taking the consequent as antecedent in relation to the antecedent as consequent.

Definition 14

A ratio *taken jointly* means taking the antecedent together with the consequent as one in relation to the consequent by itself.

Definition 15

A ratio *taken separately* means taking the excess by which the antecedent exceeds the consequent in relation to the consequent by itself.

Definition 16

Conversion of a ratio means taking the antecedent in relation to the excess by which the antecedent exceeds the consequent.

Definition 17

A ratio *ex aequali* arises when, there being several magnitudes and another set equal to them in multitude which taken two and two are in the same proportion, the first is to the last among the first magnitudes as the first is to the last among the second magnitudes. Or, in other words, it means taking the extreme terms by virtue of the removal of the intermediate terms.

Definition 18

A *perturbed proportion* arises when, there being three magnitudes and another set equal to them in multitude, antecedent is to consequent among the first magnitudes as antecedent is to consequent among the second magnitudes, while, the consequent is to a third among the first magnitudes as a third is to the antecedent among the second magnitudes.

Propositions

Proposition 1

If any number of magnitudes are each the same multiple of the same number of other magnitudes, then the sum is that multiple of the sum.

Proposition 2

If a first magnitude is the same multiple of a second that a third is of a fourth, and a fifth also is the same multiple of the second that a sixth is of the fourth, then the sum of the first and fifth also is the same multiple of the second that the sum of the third and sixth is of the fourth.

Proposition 3

If a first magnitude is the same multiple of a second that a third is of a fourth, and if equimultiples are taken of the first and third, then the magnitudes taken also are equimultiples respectively, the one of the second and the other of the fourth.

Proposition 4

If a first magnitude has to a second the same ratio as a third to a fourth, then any equimultiples whatever of the first and third also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order.

Proposition 5

If a magnitude is the same multiple of a magnitude that a subtracted part is of a subtracted part, then the remainder also is the same multiple of the remainder that the whole is of the whole.

Proposition 6

If two magnitudes are equimultiples of two magnitudes, and any magnitudes subtracted from them are equimultiples of the same, then the remainders either equal the same or are equimultiples of them.

Proposition 7

Equal magnitudes have to the same the same ratio; and the same has to equal magnitudes the same ratio.

Corollary If any magnitudes are proportional, then they are also proportional inversely.

Proposition 8

Of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.

Proposition 9

Magnitudes which have the same ratio to the same equal one another; and magnitudes to which the same has the same ratio are equal.

Proposition 10

Of magnitudes which have a ratio to the same, that which has a greater ratio is greater; and that to which the same has a greater ratio is less.

Proposition 11

Ratios which are the same with the same ratio are also the same with one another.

Proposition 12

If any number of magnitudes are proportional, then one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents.

Proposition 13

If a first magnitude has to a second the same ratio as a third to a fourth, and the third has to the fourth a greater ratio than a fifth has to a sixth, then the first also has to the second a greater ratio than the fifth to the sixth.

Proposition 14

If a first magnitude has to a second the same ratio as a third has to a fourth, and the first is greater than the third, then the second is also greater than the fourth; if equal, equal; and if less, less.

Proposition 15

Parts have the same ratio as their equimultiples.

Proposition 16

If four magnitudes are proportional, then they are also proportional alternately.

Proposition 17

If magnitudes are proportional taken jointly, then they are also proportional taken separately.

Proposition 18

If magnitudes are proportional taken separately, then they are also proportional taken jointly.

Proposition 19

If a whole is to a whole as a part subtracted is to a part subtracted, then the remainder is also to the remainder as the whole is to the whole.

Corollary. If magnitudes are proportional taken jointly, then they are also proportional in conversion.

Proposition 20

If there are three magnitudes, and others equal to them in multitude, which taken two and two are in the same ratio, and if *ex aequali* the first is greater than the third, then the fourth is also greater than the sixth; if equal, equal, and; if less, less.

Proposition 21

If there are three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them is perturbed, then, if *ex aequali* the first magnitude is greater than the third, then the fourth is also greater than the sixth; if equal, equal; and if less, less.

Proposition 22

If there are any number of magnitudes whatever, and others equal to them in multitude, which taken two and two together are in the same ratio, then they are also in the same ratio *ex aequali.*

Proposition 23

If there are three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, then they are also in the same ratio *ex aequali.*

Proposition 24

If a first magnitude has to a second the same ratio as a third has to a fourth, and also a fifth has to the second the same ratio as a sixth to the fourth, then the sum of the first and fifth has to the second the same ratio as the sum of the third and sixth has to the fourth.

Proposition 25

If four magnitudes are proportional, then the sum of the greatest and the least is greater than the sum of the remaining two.

Guide for Book V

Background on ratio and proportion

Book V covers the abstract theory of ratio and proportion. A ratio is an indication of the relative size of two magnitudes. The propositions in the following book, Book VI, are all geometric and depend on ratios, so the theory of ratios needs to be developed first. To get a better understanding of what ratios are in geometry, consider the first proposition VI.1. It states that triangles of the same height are proportional to their bases, that is to say, one triangle is to another as one base is to the other. (A proportion is simply an equality of two ratios.) A simple example is when one base is twice the other, therefore the triangle on that base is also twice the triangle on the other base. This ratio of 2:1 is fairly easy to comprehend. Indeed, any ratio equal to a ratio of two numbers is easy to comprehend. Given a proportion that says a ratio of lines equals a ratio of numbers, for instance, $A : B = 8:5$, we have two interpretations. One is that there is a shorter line C with $A = 8C$ while $B = 5C$. This interpretation is the definition of proportion that appears in Book VII. A second interpretation is that $5A = 8B$. Either interpretation will do if one of the ratios is a ratio of numbers, and if $A : B$ equals a ratio of numbers that A and B are *commensurable,* that is, both are measured by a common measure.

Many straight lines, however, are not commensurable. If A is the side of a square and B its diagonal, then A and B are not commensurable; the ratio $A : B$ is not the ratio of numbers. This fact seems to have been discovered by the Pythagoreans, perhaps Hippasus of Metapontum, some time before 400 B.C.E., a hundred years before Euclid's *Elements.*

The difficulty is one of foundations: what is an adequate definition of proportion that includes the incommensurable case? The solution is that in V.Def.5. That definition, and the whole theory of ratio and proportion in Book V, are attributed to Eudoxus of Cnidus (died. ca. 355 B.C.E.)

Summary of the propositions

The first group of propositions, 1, 2, 3, 5, and 6 only mention multitudes of magnitudes, not ratios. They each either state, or depend strongly on, a distributivity or an associativity. In the following identities, m and n refer to numbers (that is, multitudes) while letters near the end of the alphabet refer to magnitudes.

V.1. Multiplication by numbers distributes over addition of magnitudes.

$$m (x_1 + x_2 + ... + x_n) = m x_1 + m x_2 + ... + m x_n .$$

V.2. Multiplication by magnitudes distributes over addition of numbers.

$$(m + n)x = mx + nx.$$

V.3. An associativity of multiplication.

$$m(nx) = (mn)x.$$

V.5. Multiplication by numbers distributes over subtraction of magnitudes.

$$m(x - y) = mx - my.$$

V.6. Uses multiplication by magnitudes distributes over subtraction of numbers.

$$(m - n)x = mx - nx.$$

The rest of the propositions develop the theory of ratios and proportions starting with basic properties and progressively becoming more advanced.

V.4. If $w:x = y:z$, then for any numbers m and n, $mw : mx = ny : nz$.

V.7. Substitution of equals in ratios. If $x = y$, then $x:z = y:z$ and $z:x = z:y$.

V.7.Cor. Inverse proportions. If $w:x = y:z$, then $x:w = z:y$.

V.8. If $x < y$, then $x:z < y:z$ but $z:x > z:y$.

V.9. (A converse to V.7.) If $x:z = y:z$, then $x = y$. Also, if $z:x = z:y$, then $x = y$.

V.10. (A converse to V.8.) If $x:z < y:z$, then $x < y$. But if $z:x < z:y$, then $x > y$

V.11. Transitivity of equal ratios. If $u:v = w:x$ and $w:x = y:z$, then $u:v = y:z$.

V.12. If $x_1:y_1 = x_2:y_2 = \ldots = x_n:y_n$, then each of these ratios also equals the ratio $(x_1 + x_2 + \ldots + x_n) : (y_1 + y_2 + \ldots + y_n)$.

V.13. Substitution of equal ratios in inequalities of ratios. If $u:v = w:x$ and $w:x > y:z$, then $u:v > y:z$.

V.14. If $w:x = y:z$ and $w > y$, then $x > z$.

V.15. $x:y = nx:ny$.

V.16. Alternate proportions. If $w:x = y:z$, then $w:y = x:z$.

V.17. Proportional taken jointly implies proportional taken separately. If $(w + x):x = (y + z):z$, then $w:x = y:z$.

V.18. Proportional taken separately implies proportional taken jointly. (A converse to V.17.) If $w:x = y:z$, then $(w + x):x = (y + z):z$.

V.19. If $(w + x):(y + z) = w:y$, then $(w + x):(y + z) = x:z$, too.

V.19.Cor. Proportions in conversion. If $(u + v):(x + y) = v:y$, then $(u + v):(x + y) = u:x$.

V.20 is just a preliminary proposition to V.22, and V.21 is just a preliminary proposition to V.23.

V.22. Ratios *ex aequali.* If $x_1:x_2 = y_1:y_2$, $x_2:x_3 = y_2:y_3$, ... , and $x_{n-1}:x_n = y_{n-1}:y_n$, then $x_1:x_n = y_1:y_n$.

V.23. Perturbed ratios *ex aequali.* If $u:v = y:z$ and $v:w = x:y$, then $u:w = x:z$.

V.24. If $u:v = w:x$ and $y:v = z:x$, then $(u + y):v = (w + z):x$.

V.25. If $w:x = y:z$ and w is the greatest of the four magnitudes while z is the least, then $w + z > x + y$.

Logical structure of Book V

Book V is on the foundations of ratios and proportions and in no way depends on any of the previous Books. Book VI contains the propositions on plane geometry that depend on ratios, and the proofs there frequently depend on the results in Book V. Also Book X on irrational lines and the books on solid geometry, XI through XIII, discuss ratios and depend on Book V. The books on number theory, VII through IX, do not directly depend on Book V since there is a different definition for ratios of numbers.

Although Euclid is fairly careful to prove the results on ratios that he uses later, there are some that he didn't notice he used, for instance, the law of trichotomy for ratios. These are described in the Guides to definitions V.Def.4 through V.Def.7.

* Some of the propositions in Book V require treating definition V.Def.4 as an axiom of comparison. One side of the law of trichotomy for ratios depends on it as well as propositions 8, 9, 14, 16, 21, 23, and 25. Some of Euclid's proofs of the remaining propositions rely on these propositions, but alternate proofs that don't depend on an axiom of comparison can be given for them.

Propositions 1, 2, 7, 11, and 13 are proved without invoking other propositions. There are moderately long chains of deductions, but not so long as those in Book I.

The first six propositions excepting 4 have to do with arithmetic of magnitudes and build on the Common Notions. The next group of propositions, 4 and 7 through 15, use the earlier propositions and defintions 4 through 7 to develop the more basic properties of ratios. And the last 10 propositions depend on most of the preceeding ones to develop advanced properties.

Dependencies within Book V	
2	3, 6
3	4
1	5, 8*, 12
8*	9*
7, 8*	10
8*, 10, 13	14*
7, 12	15
11, 14*, 15	16*
1, 2	17
11, 14*, 17	18
11, 16*, 17	19
7.Cor, 8, 10, 13	20, 21*
4, 20	22
11, 15, 16*, 21*	23*
7.Cor, 18, 22	24
7, 11, (14), 19	25*

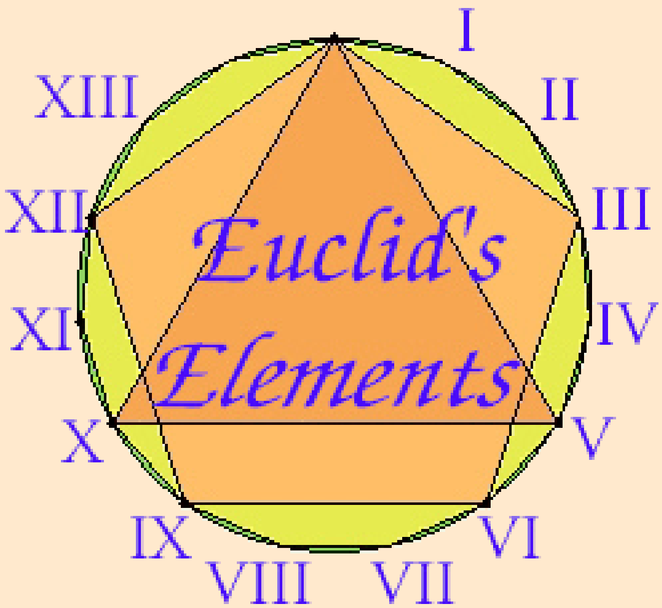
Next book: Book VI

Previous: Book IV

Select from Book V

Select book

Select topic



Book VI

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Definitions

Definition 1.

Similar rectilinear figures are such as have their angles severally equal and the sides about the equal angles proportional.

Definition 2.

Two figures are *reciprocally related* when the sides about corresponding angles are reciprocally proportional.

Definition 3.

A straight line is said to have been *cut in extreme and mean ratio*

Select from Book VI

Select book

Select topic