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Whitehead, Alfred North, 1861-1947., Russell, Bertrand, joint author. 1872-1970.

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PRINCIPIA MATHEM4ATICA

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Page III

PRINCIPIA MATHEMATICA BY ALFRED NORTH WHITEHEAD, Sc.D., F.R.S. Fellow and late Lecturer of Trinity College, - Cambridge AND BERTRAND RUSSELL, M.A., F.R.S. Lecturer and late Fellow of Trinity College, Cambridge VOLUME III Cambridge at the University Press 1913

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PREFACE TO VOLUME III THE present volume continues the theory of series begun in Volume II, and then proceeds to the theory of measurement. Geometry we have found it necessary to reserve for a separate final volume. In the theory of well-ordered series and compact series, we have followed Cantor closely, except in dealing with Zermelo's theorem (*257-8), and in cases where Cantor's work tacitly assumes the multiplicative axiom. Thus what novelty there is, is in the main negative. In particular, the multiplicative axiom is required in all known proofs of the fundamental proposition that the limit of a progression of ordinals of the second class (i.e. applicable to series whose fields have 0o terms) is an ordinal of the second class (cf. *265). In consequence of this fact, a very large part of the recognized theory of transfinite ordinals must be considered doubtful. Part VI, on the theory of ratio and measurement, on the other hand, is new, though it is a development of the method initiated in Euclid Book V and continued by Burali-Forti*. Among other points in our treatment of quantity to which we wish to draw attention we may mention the following. (1) We regard our quantities as in a generalized sense "vectors," and therefore we regard ratios as holding between relations. (2) The hypothesis that the vectors concerned in any context form a group, which has generally been made prominent in such investigations, sinks with us into a very subordinate position, being sometimes not verified at all, and at other times a consequence of other more fruitful hypotheses. (3) We have developed a theory of ratios and real numbers which is prior to our theory of measurement, and yet is not purely arithmetical, i.e. does not treat ratios as mere couples of integers, but as relations between actual quantities such as two distances or two periods of time. (4) In our theory of "vector families," which are families of the kind to which some form of measurement is * Cf. Peano's Formulaire, i. (1895), pp. 28-57. 252817 vi PR EFACE applicable, we have been able to develop a very large part of their properties before introducing numbers; thus the theory of measurement results from the combination of two other theories, one a pure arithmetic of ratios and real numbers without reference to vectors, the other a pure theory of vectors without reference to ratios or real numbers. (5) With a view to geometrical applications, we have devoted a special Section to cyclic families, such as the angles about a given point in a given plane. The theory of measurement developed in Part VI will be required in the next volume for the introduction of coordinates in Geometry. We have to thank various friends for their kindness in bringing to our notice mistakes and misprints noted in the Errata, both in this and in previous volumes. A. N. W. B. R. 15 February 1913

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ERRATA TO VOLUME III p. 3, line 22, for "that there is" read "that this is." p. 23, *251-371, add "[*251 37. *170-101. *171101]." p. 25, lines 2 and 3,for " 183-11 and *185-11 " read "*183'14 and *185'12." p. 25, line 6, for "[*251'61. *183-11. *185'11]" read " Dem. F. *151'65.*182'05'162.).. C'P e(J(;P) smnor P n Rl'smor (1) F. (1). *15116. D: Hp. D.! {(P)si6r($; Q)} n Rl'smor (2) F. (1). 251111. *182-16. D F: Hp.. C';iP C Q.;P,; Q e Rel excl (3) F. (2). (3). *251-62. *18314. *185-12. D. Prop" p. 33, line 6, for "/3 8 w " read "w x /3." p. 44, line 12, for first " P " read " Ps." p. 59, line 20, for "*120-413" read "*120'51." p. 71, line 6 from below, for " Nr'C'P" read "Nc'C'P." -- -- p. 89, line 4 from below, for "Hp. z e QR;'y" read " Hp. ye a. z e QZy." p. 90, *257-211, first line of Dem.,for " Hp." read " Hp.. ". p. 99, *258'221, for "p'c = (R*Q)'c" read "p'c = limax (QRa)K." p. 112, line 8, bor " xP1z" read " xPz." p. 120, line 9 from below, for "a well-ordered" read "an infinite well-ordered." p. 177, *265-48-49,for "Itp'Q" read "Itp'C'Q." p. 194, *272-161, second line of Dem., for "z eD"" read "Hp.z e D'T." p. 195, *272-221, add " z, we D'T" to hypothesis. p. 196, *272321, for " z" read " w.",,,TQP,, p. 198, *272-42, second line of Dem.,for ' " read T,, Q' T, P, Q p. 210, *274'12, fourth line of Demn.,for "(az)" read "(a[z). z e a -." p. 217, *274'31, for " P Ser n comp " read " P, e Ser n comp." 4 -p. 224, *276-2, Dem., line 4, for " y = / v Pz " read " y = (a n P*'z) v P'minp'(a n P'z)." p. 224, *276-2, Dem., line 5, for " y n P'z = 8 n Pz = a n Pz. z e a - " 4- - 4- - 4 --- read " minp'(a n P'z) e a - y. a A P'minp'(a n P'z) = y n P'minp'(a er P'z). z e y- B.7 n P'z = a n P'z =, P'z." p. 224, *276'2, Dem., line 6, delete " *170'16." p. 244, line 12, for "(I R) I (-o 1)" read " [{(R) 1(-, 1)}." p. 316, line 15,for "limit" read "limit or maximum."

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x ERRATA p. 317, line 15, end, add "including zero." p. 320, last line but one, for "rationals" read "real numbers." p. 347, line 9 from below, for first " D " read " I." p. 347, line 8 from below, for " x + y = x + z" read " y + x = z + x." p. 379, *333-24, enunciation, for "! v n t2'R " read " a! (v +c 1) A t2'P." p. 379, line 8 from below, for "! v n t'R is e C' U t3'R " read " a! (v +~ 1) n t2'P is v e (' U t3'P." p. 379, *333-25, enunciation, for "! v n t2'L " read " f! (v +c 1) n t2'L." p. 395, line 14,for "greater and less" read "greater and less among magnitudes." p. 433, *353-22, line 4 of Dem., for " Hp " read " Hp. R e X." ADDITIONAL ERRATUM TO VOLUME I p. 574, line 8,for "p'a"(S R)" read "pC'"Pot,(s R)." ADDITIONAL ERRATA TO VOLUME II p. 11, line 13 from below, for " Nc'a n to"'a" read " Nc'a n t'to'a." p. 71, last line, for " Nc'aNc' = N0c'aNoc'I Df" read "(Nc'a)r = (Noc'ca) Df and UNc' o = pNoc'c Df." p. 90, line 21,for "mutually exclusive classes of,u " read " classes of p mutually exclusive." p. 152, margin of figure, for " D'R = D'M'z " read " D'R = D'Mw." p. 152, last line, for "ea'T"7y" read " e'T"7y." p. 279, line 11, for " *124234 " read " *124-23'25'252." p. 334, line 10 from below, for "series" read "well-ordered series." p. 347, line 7 from below, for "the series a" read "the class a." p. 348, line 5 from below, for "Cls3 excl" read "Cls2 excl." p. 366, line 8, for " Re C'Q " read "M e CQ." p. 366, line 9, for " such relations as M" read "such relations as R;M." p. 403, footnote, for "mathematischphysischen" read " mathematisch-physischen." p. 519, *200-21, for " Te Cls -- 1" read " Te Cls -- 1. P J." p. 561, last line, for "D" read " a." p. 570, first line, delete "which is used in *263'11." p. 606, line 23, for "*208-4 " read " *208-41." p. 606, line 24, for "S, e P gsmo Q. i. S = T" read "P smor Q.). (P smTor Q) e 1." p. 614, line 7,for " *214-31" read " *214'32." p. 710, Summary, line 2,for "P" read "Q." p. 710, Summary, line 3, for "Q" read " P." p. 753, footnote, for "reallen" read "reellen."

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SECTION D. WELL-ORDERED SERIES. Summary of Section D. A " well-ordered " series is one which is such that every existent class contained in it has a first term, or, what comes to the same thing, one which is such that every class which has successors has a sequent. We will call a relation in general well-ordered if every existent class contained in its field has one or more minima. Then a well-ordered series is a series which is a well-ordered relation. Well-ordered series have many important properties not possessed by series in general. A well-ordered series is Dedekindian, except for the fact that it may have no last term; i. e. every section having a last term is Dedekindian. A well-ordered series which is not null has a first term, and every term of the series (except the last, if there is one) has an immediate successor. A very important property of well-ordered series is that they obey an extended form of mathematical induction, which we shall call "transfinite induction," namely the following: If a is a class such that the sequent (if any) of any class contained in a and in the series is a member of a, then the whole series is contained in a-. (It will be observed that A is contained in a-, and therefore, by *206'14, B'P is a member of a.) This differs from ordinary mathematical induction by the fact that, instead of dealing with the successors of single terms, it deals with the successors of classes. A closely analogous property, which holds for all well-ordered relations, whether serial or not, is the following: If a is a class such that, whenever P'x C a, where x is any member of C'P, x itself belongs to a, then C'P C a. If P is well-ordered, this property holds for all a's; and conversely, if this property holds for all a's, P is well-ordered. Hence this property is equivalent to well-orderedness. If P is a well-ordered series, minp selects one term out of each member of Clex'C'P. Hence C'P, which is minp"Clex'C'P, is a member of the multiplicative class of Cl ex'C'P; hence the multiplicative class of Cl ex'C'P exists, and therefore the multiplicative class of any class contained in C1 ex'C'P exists (by *88-22). It follows that if s'i can be well-ordered, and A e i, the multiplicative class of K exists; and that, if every class can be R & w. IIL 1

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2 SERIES [PART V well-ordered, the multiplicative axiom holds. The converse of this latter proposition also holds, as has been proved by Zermelo (cf. *258). Another important set of properties of well-ordered series results from *208'41 ff. Two ordinally similar well-ordered series can only be correlated in one way; and no proper section of a well-ordered series is ordinally similar to the whole series. (A "'proper'" section is a section not the whole.) From the uniqueness of the correlator of two similar well-ordered series, it follows that all the uses of the multiplicative axiom in *164 can be avoided if the fields of the relations concerned consist of well-ordered series. I.e. taking *164'45, which is the fundamental proposition in this subject, we have, without assuming the multiplicative axiom, P, Q e Rels excl. 3: a! P smior Q r Rl'smor. _. P smor smor Q, whenever C'P and C'Q consist of well-ordered series. Hence, under this hypothesis, the multiplicative axiom disappears from the hypotheses of all the consequences of *164*45. Ordinal numbers (*251) are defined as the relation-numbers of wellordered series. (This definition is in accordance with usage: otherwise, there would be no special reason against defining "ordinal numbers" as the relation-numbers of series in general. The relation-numbers of series will be called serial numbers.) Sums of an ordinal number of ordinal numbers are ordinal numbers, but products of an ordinal number of ordinal numbers are not in general ordinal numbers. The product of an ordinal number of serial numbers is a serial number, and the product of an ordinal number (not zero) of ordinal numbers other than zero is not zero, i.e. a product of ordinal numbers, in which the number of factors is an ordinal number, does not vanish unless one of the factors vanishes. (For relations in general, the corresponding proposition requires the multiplicative axiom.) If v is an ordinal number, and, is any serial number, lkexprv (i.e. /p as it would naturally be called) is a serial number; but if p > 1, 1A expr v is not an ordinal number unless v is finite. The theory of sections and segments (*252, *253) is much simplified for well-ordered series, owing to the fact that every proper section has a sequent. Proper sections are identical with proper segments, and both are identical with P"C'P. The series of sections, s'P., is P;P++C'P. The series of segments, s'P, is P;P or P;P-. UC'P according as there is or is not a last term of 0'P. The series of sectional relations, Ps, is P t;P;'P C(IP- P; its domain is P "P"C'P, and its field is. P "P"C'P v lP. If eC 'P, P P P'x is never similar to P.

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SECTION D] WELL-ORDERED SERIES 3 The theory of greater and less among well-ordered series and ordinal numbers is dealt with in *254 and *255. Cantor has proved, by means of segments, that of any two different ordinal numbers one must be the greater. This is proved by showing that of any two well-ordered series which are not similar, one must be similar to a segment of the other. We define an ordinal number a as less than another / if series P and Q can be found such that P is an a and Q is a / and P is similar to some relation contained in Q, but not to Q. It can be proved that all the ordinals less than Nr'Q belong, one each, to the proper segments of Q. Hence to say that the ordinal number of P is less than that of Q is equivalent to saying that there is a proper segment of Q to which P is similar. When two series have the same ordinal, they also have the same cardinal, in virtue of *151'18, but the converse does not hold. When the cardinal number of one series is greater than that of the other, so is the ordinal number. When two classes can be well-ordered, any well-ordering will make the one class similar to a part of the other, or the other similar to a part of the one, in virtue of the properties of segments of well-ordered series. Hence of two different cardinals each of which is applicable to classes which can be well-ordered, one must be the greater-a property which cannot be proved concerning cardinals in general. In *256 we deal with the series of ordinals in order of magnitude. We show that there is a well-ordered series, and that the series of all ordinals of a given type has an ordinal number which is greater than any of the ordinals of the given type. This constitutes the solution of Burali-Forti's paradox concerning the greatest ordinal: there is no greatest ordinal in any one type, and all the ordinals of a given type are surpassed by ordinals of higher types. *257, *258 and *259 deal with "transfinite induction" and its applications, of which the most important is Zermelo's theorem, namely, *258-34. ':. 6 e 1.: Se a'Cl ex'. -. (aP). P e i. C'P =. S = minp r C1 ex'l where QI is the class of well-ordered series. This proposition leads to the following: *258-36. F:, e C"n u 1. =. a! eA'C1 ex' I.e. a class can be well-ordered or is a unit class when, and only when, a selection can be made from its existent sub-classes. Hence we arrive at *258-37. F: Mult ax. =. C"11 v 1 = Cls I.e. the multiplicative axiom is equivalent to the assumption that every class can be well-ordered or consists of a single member. The proof of Zermelo's theorem uses an extension to transfinite induction of the ideas of *90 and *91, which is explained in *257. 1-2

*250. ELEMENTARY PROPERTIES OF WELL-ORDERED SERIES. Summary of *250. A relation is called " well-ordered" when every existent sub-class of its field has one or more minima. A well-ordered series is defined as a wellordered relation which is a series. We shall denote the class of well-ordered relations by " Bord," which is an abbreviation for " bene ordinata " or "bien ordonn6e." The class of well-ordered series will be denoted by 12. Thus our definitions are Bord = P (Cl ex'C'P C (minp) Df, 1 = Ser n Bord Df. Well-ordered relations other than series will be seldom referred to after the present number. By applying the definition of " Bord" to unit classes, it appears that a well-ordered relation must be contained in diversity (*250'104). A wellordered relation is one whose existent upper sections all have minima (*250-102). Hence by *211-17, *250103. F: P e Bord. =. Pp e Bord Hence by *250-104, *250-105. F: Pe Bord. 1. Ppo C J By considering couples, it can be shown (*250'111) that a well-ordered relation in which no class has more than one minimum is connected; hence by *204*16 and *250'105, it is a series. Thus we have *250'125.: P e. _. E!! minp"Cl ex'C'P, I.e. a well-ordered series is a relation such that every existent sub-class of the field has a unique minimum. This might have been taken as the definition of fl. By the definition of 1f we have *250-121. h:. P e. -: Pe Ser: a C C'P.! a. 3D. E! minpla:: Pe Ser: a! an C'P. 3,. E! minp'a Applying this to C'P we have *260'13.: P e f - tA. D. E! B'P

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SECTION D] ELEMENTARY PROPERTIES OF WELL-ORDERED SERIES 5 We have also *250-141.: Pe f. D. P a e f1 *25017.:. P, Q e - 'A.: P smorQ.-. P (I'P smor Q ('Q This proposition justifies the subtraction of i from the beginning, and is useful in the theory of segments of well-ordered series. We have next (*250'2 —243) an important set of propositions on P1 when P e f. The most useful of these is *250'21. F: P e f1. 3). D'P = D'PI I.e. in a well-ordered series every term except the last (if any) has an immediate successor. (It is not in general the case that every term except the first has an immediate predecessor.) Another useful proposition is *250-242. F: Pe f. D. P = Pil P IP The next set of propositions (*250'3 —362) is concerned with "transfinite induction." We have *250'33. F. 2 = connex n P {a C 'P n. a. seqp'a C o:),. C'P C o I.e. a well-ordered series is a connected relation P such that the whole field of P is contained in every class a which is such that the sequent (if any) of every sub-class of C'P n a- is a member of ar. *250 35.. Bord = P x e 'P. P'x C ao.). x e a:). C'P C ao I.e. a well-ordered relation is a relation P whose field is contained in every class a- which contains every member of C'P whose predecessors are all contained in a-. We may say that a property is " transfinitely hereditary" in P if it belongs to the sequents of all classes composed of members of C'P which possess the property. In virtue of *250-33, if P is well-ordered, every transfinitely hereditary property belongs to every member of C'P, and conversely. Our next set of propositions (*250'4-'44) is concerned with A and couples. We prove that Ae f (*250-4) and that x y..x ye f (*250'41). *250'5 —54 are concerned with selections. We have *250-5.: Pe.. minp I Cl ex'C'P e e'Cl ex'C'P. t'C'P = Prod'Cl ex'C'P whence *250-51. F: a e C1. D. a! e'Cl ex''a Observe that C"12 is the class of those classes that can be well-ordered. From *250'51 we deduce *250-54. F: C"f u 1 = Cls.. Mult ax The converse, which is Zermelo's theorem, is proved in *258.

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6 SERIES [PART V *250-6 —67 are concerned with consequences of *208. We show that two well-ordered series cannot have more than one correlator (*250-6); that if P is a well-ordered series, and 8 is contained in a proper section of P, P

C / is not similar to P (*250-65); and that if P is any well-ordered relation, and a is any class such that there are terms in C'P which are later than any member of a n C'P, then P is not similar to P e a (*250-67). *250-01. Bord = P (Cl exP'CP C Pminp) Df *250-02. fl = Ser e Bord Df *250-1. F: P e Bord. =. Cl ex'C'P C G'minp [(*250-01)] *250-101. F:. Pe Bord. a! C'P. D.. a! minp'a [*250-1 *205-15] *250-102. F P e Bord..sect'P - V'A C GI'minp Dem. F.*250-1.) F: PeBord.).sect'P-tffAC I minp (1) F. *2019.) F..min (P0)'a = min (P,)'P*"a [*205-68] = miP'P"*"a (2) F. *901331.*21113 3.F: a n C'P.. P"a e sect'P - t'A (3) F.(3).) F:. sect'P - t'A C U'Ininp.): a! a n C'P. a!. a! minp'(P*"(a). [(2)] min(P1)'a. [*205-26]).!. a!minp'a: [*250-101]): Pe Bord (4) F.(1).-(4).) F.Prop *250-103. F: P e Bord.. Pp e Bord [*250-102. *211-17] *250-104. F. Bord C Rl'J Dem. F.*250'1.)F Pe Bord. xe C'P. D.xe minpL':x. [*205-194] D., (xPx):) F. Prop *250105. F: P e Bord.) PP J [*250-103-104] *250-11. F:: Pe connex.D:. Pe Bord. a:!an C'P. D.. EIminp'a: 2:2a C G'P. a! a.:),. E I minp'a [*250~1101. *20532] *250-111. F:. P e Bord.: Peconnex.2. minp el -+ Cls Dem. F.c*250-1.*771.1 F::P e Bord.minp elI -+ Cis.):. X, y e C'P. D: ('lx v t'y) - P"(ffx u t'y)e l: [*54-4]: txuy - P"(ffL'xu t'y)- L'=x. V. t'xuv t'y -PV (It'xi'Y)=-'1Y (1)

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 I-.(1)_.: F:. P e Bord. mlinpe Il-* CIS. x, ye C'P. x~y.:: y e P"6(fx v t'ly). V.* x e P"6(t'x v y) [*250-104]:):Xpy.v.yPx (2) F.(2). *202-103.) F:P eBord.minp elI —.CIs.)P cconnex (3) F. (3). *205 31 F)I-.Prop *250-112. F:P econnex nBord.. E!! minp"Cl cex'C'P Dem. F.*250- 1 11. F- P e connex n Bord..Minp 1 -.) CIS. Cl ex'G'IP C PIminp. [*71-16] E.1 El!infl"flpGIniin~p. Cl exP'CP C U'1miolp. [*205-15-16].E! rniinp"'CI exP'P:)F. Prop *250-113. F. connex r^ Bord =l Dem. F. *204-1. (*250-02).)F. f2 C connex nBord (1) F.-*250-105.) F: —P econnex riBord.)P econnex. PP f. J [*204-16]).P e Ser (2) F. (2). (*250-02).) F:.Pe connex riBord.)P e f (3) F.-(1). (3).DF..Prop *250-12. F:P El2.=. P ESer rtBord [(*250-02)] *250-121. F:. Pe =-: Pe Ser:a CG'P. [!a.)..E!minp'a: -:PeSer:g! avnCGP.)a'.Elminp'la [*250-12711] *250-122. F:.Pef~.=E:PeSer:f! C'Pnp'P",(anrC'P).:)4.E!seqp'c Dem. F. *206-13. *250-121. F:.PefL.):PeSer::4!C'Prtp'P"6(anCIP).:),,.E!seqp'a (1) F.*204-62.) F:P eSer. a!a n C'P. a! CP ^p'P6,p'P",(avYI'P). [*40-62]).! C'Pf npP"{ffC'Pnp'P"((arnG'P)J (2) F. (2). *10-1.) F:.P e Ser:a! C'P p,'P" (a n C'P). 3.. El! se(lp'a:: a!a n CP'PoD. aE! seqp'{G'CP ri p'P"l(ari% C1P)}. Lr*206-13154]).E! minpla: [*250-121])D: P e f (3) F. (1).(3).:)F. Prop

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 *250-123. F:Pf-tA=:eOr'pP,(nlP).:).Elseqp'a Dem. F. *250-122. D 4 -F:.P eSer: a! p'P",(an C'P).,..E! seqp'a: D. P ef~ (1) F. *40-6. *24-52. D F:. aj!p'P"1(a n C'P).D)., E I seqp'a: D. E! seqp'A. [*206-18].!P (2) F *250-122. *40-62.)D F. *206-14.) 1-F a n C'P = A.). seqp'a = BIP [*205-12] = minp'G'P (4) F. *33-24. *250-121. DF:P efl - tA.D. E! ninp'C'P (5 F.:4.Pe5)'.)F:Pefer-jt'AaCP=

(ACP.)D.E!seqp'la (7) F.(1).(2). (7). D F. Prop *250-124. F: Pe2 f. M. Pe Ser. sect1P - 'C'P C (lseqp Dem. F.*250-122.*211 703.)F-:PEfl.).P~eSer.sect'P-tP'CPCG'seqp (1) F *211-7. D)F:P eSer. sect'P - ffC'P C (1seqp. D 13 sect'P - t A.), p. E! seqp'(C'P - i3).[*211 723] Dp. El! minp'13: [*250-102,12] D: P 6 f (2) F. (1). (2). D F. Prop *250-125. F: P e 2 =- EN minp"'Cl ex'C'P [*250-112-113] The above proposition might be demonstrated, independently of *250-112-113, as follows: (a) If E I!ninpl"Cl ex'C'P, it follows that x eC'1P.:). Elminp'L'1x, whence x eC'1P.:). -(xPx), whence PC- J. (b) If E! minp,"Cl ex'C'P,it follows that X, y eC'P -x +y.). E I minp'(txa vul) whence it follows that XPy.(yPX). V. yPX. e-.' (XPy). Hence P econnex. P2C- J. (c) If E I! minp,"Cl ex'C'.P, it follows that xy yPz. D). EII minp'(t'x u tu ~)

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SECTION D] ELEMENTARY PROPERTIES OF WELL-ORDERED SERIES 9 whence xPy. yPz. D. e-(zPX), and by P2 C J (which has just been proved) xPy. yPz. 2. x + z. Hence, since, by (b), P e connex, we must have xPy. yPz.:).xPz, i.e. Pe trans. Hence E!! minp"'C1 ex'G"'P. 2. Pc Ser. Hence the above proposition is obvious. *250-126. F:Pef1.E!maxp'a.. E iseqp'l.2).B'Pea.B'P=maxp'a Dem. 1.*2250'123. Transp. 21-: Hp. 2.%j!p'Pt(ancc'P). 4 -[*205-65] 2. U! P'maxp'a. [*33'4] 2. maxp'a e D'P. [*93-103] 2. maxp'l a B'P. [*202-52] 2. maxp'l = B'P: 2 F. Prop *250-13. F: P e 2 -'. 2.E! B'P Dem. F. *33-24. 2F: Hp. 2. a! U'P. [*250-121]. E! minp'C'P. [*205-12] 2.E! B'P: DF. Prop *250-131. F:. PcfL 2.2rD P. E! B'P Dem. F. *93-102. *33-24. 2 F: E! B'P. I P (1) F. (1).*250-13.DF.- Prop *250-14. F: P E Bord. 2. R1'P C Bord Dem. 1-.*2501. *205-26. D F: Pe Bord. Q C P.. C1ex'C'P C (I'minP. minp r C1 ex"'Q CG minQ. (1) [*60-42.*3.564] 2.01 ex'Q C Cl exPC'P. G'minp n Cl ex'C'Q C U'minQ (2) F.(1). (2). *22-44621. 2 F: P e Bord. Q G P. 2. Cl ex'C'Q C GI'minQ. [*250-1] 2. Q e Bord: D F. Prop *250-141. F:PefZ.2.P Cae f [*250-14. *204A4] *250142. F: Pe Bord. 2. R1'P n connex C f Dem. F. *250-14. 2 F: Hp. 2. RI'P n connex C Bord n connex [*250-113] Cr2I: D. Prop

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10 SERIES [PART V *250'15. I-:Pefl.E!B'P.. PeDed Dem. F. *250-101. 3:. Hp.): g! a n 'P..! minp'a (1) F. *206'14. F:.Hp.: a C'P = A.). ^.! prep'a (2) F. (1). (2).) F: Hp.]. (a). P! (minpa u precp'a). [*214-1] Z. Pe Ded. [*214-14] D. P eDed: F. Prop *250-151. F: Pe n. x e P.. P. P P*'x e Ded Dem. F. *250-141.): Hp. p. Pt*'x e (1) -. *205,41.) I-: Hp.). B'Cnv'(P t P,'x)= maxp'P*'x [*205-197] = x. [*53'3]). E! B'Cnv'(P t PC'x) (2) F.(1).(2).*25015.)F. Prop *250'152. F. f C semi Ded [*2147. *250-124] *250'16.: P e. a n CP.. P'minp'a =p'P"(a n CP) [*205-65. *250-121] *250-17. F:. P,Q e - LtA. Z: P smor Q. -. P C a(P smor Q (PQ [*204-47. *250-13] This proposition is useful in connection with the series of segmental relations in a well-ordered series, for the series of proper segmental relations in a well-ordered series is (as will be proved later) P;P;P eC a P, and this is ordinally similar to P (P'P. Hence, by the above proposition, two well-ordered series which

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are not null are ordinally similar when, and only when, the series of their segmental relations are ordinally similar. *250-2. F: P e Bord.. D'P = D'(P-P 2) Dem. 4 -F. *33'4. F:ax eD'P..!P' (1) 4- -44 -F-. *250'1. *205'16. D F:. P Bord. D:a! P 'x. g minp'P'x. [*205251]. -. xe D'(P -P2) (2) -.(1). (2). 2-F. Prop

SECTION D] ELEMENTARY PROPERTIES OF WELL-ORDERED SERIES 1 11 *250-21. F:P efl.).D'P =D,'Pi [*201-63.*250-2] In virtue of this proposition, every term of a well-ordered series (except the last, if any) has an immediate successor. *250-22. F:PeSernDed.D'P=D'P1.:). Pefi~-ffA Dem. F.*214-101.)F:Hp.,-.,E! maxp'ca E!seqp'a (1) F.*'206-45.:)F:Hp.maxp'lae D'P.). El!seqp'~maxp'a[*206-46] D. E!seqp'a (2) [*93-118] D:r,,(B'Pea).D..E!seqp'a: [*202-511.*2145a]):a! p'P"1(c rn C'P).). E! seqp'a: [*250-123] D:PEI 2 - t'A:. DF.Prop *250-23. F:Pefl.E! B'P.=-.PeSerrtDed.D,'P=D'PI Dem. F.*250-22. *214-5.:)F:P Ser n Ded.DIP = D'P1.. P efl. E!B'P (1) F. *250-1521. D F:Pef2. E!BP.).P eSer ^Ded.D'P D'PI (2) F. (1). (2). D F. Prop *250-24. F: P ef2.:.P2 (PI = P D'P Demn. F. *201-1.*13-12.) DF:. Hp. xP2z. D: yPx.2D.yP2z: y= x)yP2z: [*201P63.*202-103])yP'z.:. xPy (1) F. (1). *201-63. D F: Hp. xP2z. zP~y. D. xPy. x, y eDIP (2) F.*250-21. D)F: Hp. x,ye DIP. xPy.).(Hz).yP~z [*201-63] D. (HZ).yPz. zP'y. F (2).(3).:)F.Prop *250-241. F: P e fi. 2. PI =2 (jI'Pj) P [Proof as in *250-24] *250-242. F: P e fL.D.P=PI PII P Dem. F. *201-63.DF:: Hp.):XPY n: Xply. V. XP2y: [*250-21] =:xP'y.v.(Hz).xPiz. xP2y: [*250-241]:XPI Y v.(z.xPIZ. zPY: 2 Prop

12 SERIES [PART V *250-243. F: P ef.D. P IPi = (,PP) 1(P, v PJI P,) [Proof as in *250-242] The following propositions deal with the extended form of mathematical induction which is characteristic of well-ordered series. *250-3. F:. P e Bord: a C C'P n a-. D.2 seqp'aCa-: D. CG'P Ca Dem. F. *250-101. 2 F: P e Bord. a! C'P - a. 2.! minp'(C'P - a). [*205-14] 2. (ax). x c C'P - a-. P'x: C a-. [*206-4.*250-104]2D. (ax). x e C'P - a. P'x C a. x seqp (P'~x). [*13-195] D. (Hx, a). a = P'x. a C C'P n a-. xe seqp'a - a. [*r10-24] D. (Ha). a C CIP n o-. a! seqpla - a 1 F. (1). Transp. 3.FProp *250-301. F: P econnex.e' 3! minp'z-. a- = C'P-PP"'r. aC a-.. seqp'aCaDem. CF. *205-122. *202-501. DF:Hp. D. a-CCp'P T. [*40-67] 2.i-Cp'P,a- (1) -4 F. 4E206-134. F: Hp. x seqp a. D 2 P"X C - pIPc'a [*40-16] C-P P ta[(1)] C-r. [*37-462] 2. XI', P" 7. [*206-18.Hp] D.xe-: 2F. Prop *250-31. F:: P e connex:.a C C'P n a-. D.seqp'a C oa-:2. CP C a:..P e f Dem. F. *250-301.2 F:. P e connex - a! C'P n - '7-. —a! minp"7r. a-= C'P - P"7-. 2: a C a-. D..seqp'a C a!C'P-u (1) F. (1). *10-28. D F:. P e connex: (a7-) a a! C'P n T. a! minp'7r: 2: (a-): a C a. D.. seqp'a C a: C'P - a (2) F. (2). Transp. D -4 F:: Pce connex:. a Ca. D.. seqp'a Ca-: 2,. ('P C a-:. 2: a! C'P A 2. 7- I minplr: [*250101] D: P c Bord (3) F.(3).*250-113. 2. Prop

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-14 SERIES [PART V [*206-171] x = seqp'(a- %P'x). [(2)] a.! c %P'x.P'xa P.xC {seqsp'(o- fP'x) C o-' I-.(4). Tranasp. F:Hp. E!- El inp'(P"ao- - a-). [(1).Transp]) P~a- - a- =A: D F- Prop *250-361. F:. P e f2.Pl"o-C -:XC a-.a! (X rt CP). D,.1imaxp'X C -: D. p66 T C aDent. F. *206A46-43.:) F: Hp. X C a -. E! maxp'X. D. seqp,'X = P1rniaxp,'X. [Hp] seqseqp'X o- (1) I- *20A.~F:. Hp.C o:~-.-! X n(J'P) El) max -s'pX..sq'X Imx: [*250-36] PI"o- C a-:.)D F. Prop *250-362. F:.Pefl. Pl"oCo-:,XCa-.3!Xr~.'P.:)A.1iminp'XCo-::)..P"ao- C aL 250-361. *121.6 *250-4. F. Ae fl Dem. F. *60-33.)F. Cl ex'C'A C PImin (A) (1) F. (1). *250-1.)DF.A eBord (2) F.-(2). *204-24.)D F. Prop *250-41. F:xoy.D.xlyef2 Dem. F. *60-39. D F.Cl exP'C (x 4, y) = ul~ ll v ff,(t'x v c',y) (1) F*205,18.)F:Hp. P x 4,y.) minp"x = x. minp"tly-y (2) F.*205-181. D F: Hp (2). m i mnp',(t'x v t+y6=x (3) F.(-1>. (2). (3) D - Hp (2)).Cl ex'G'(x 4,y) C PIlminp. [*250-1] D. x4y eBord (4) F. (4). *204-25. D F.Prop

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ex'C'P. L'C'P=Prod'Clex'C'P [*20533. *25`01. *115-17] This proposition is of great importance, since it gives the existencetheorem for selections from any class of existent classes whose sum can be well-ordered (cf. *250-53, below). Observe that "a e CIf2" means "cfa is a class which can be well-ordered." *250-51. F: a C G"If.).!. e4'Cl ex'a [*250-5] *250-52. F: ae EC. j9Ca.:).l! E4'Cl ex' [*88&222. *250-51] *250-53. F:s'KeG"172.Ar..E.).a!E&'x Derm. F.*60'23~57.)F: Hp.). tcC Cl ex's'#c. [*88&22.*250-51] a! e4x:DF. Prop *250-54. F:CG"QfuvIl-Cls.).Multax Dem. F. *250-53.*834.):.FHp. D: Ae.K.)e a! E6'rc: [*88&37] M:I/ult ax:F.)hProp

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 The above proposition states that if every class which is not a unit class is the field of some well-ordered series, then the multiplicative axiom holds. The converse of this proposition has been proved by Zermelo (cf. *258'47). *260-6. F: P, Q e f. PsmorQ.. Ps Q e 1 [*208'41. *25012-1] This proposition is very useful, since it enables us, when two similar series of similar well-ordered series are given, to pick out the correlators of all the pairs without assuming the multiplicative axiom. I.e. given P, Q e Re excl. S e P sinor Q. S smor, if N e C'Q, the correlator of S'N and N will be t'(S'N) smor N if S'N, Ne &. This enables us to dispense with the multiplicative axiom in the hypotheses of *164'44 and its consequences, whenever the relations concerned have fields whose members are well-ordered series. *250*61.: P e.. P smor P = t'(I r C'P) [*208-42] *260-62. F: P e Bord. S e cror'P.).. (gax). (S'x) Px [*208-43] *250 63. F: P e f n Cnv".:). RI'P n Nr'P = t'P [*208'45] This proposition will be useful in showing that a finite series is not similar to any proper part of itself, and is a series which is well-ordered and has a converse which is also well-ordered. *250'64. F: P e Bord. S e cror'P. D. C'P n p'P"D'S = A [*208-46] In virtue of this proposition, a part of a well-ordered series can only be similar to the whole if the part extends to the end of the series. Thus e.g. no proper section of a well-ordered series can be similar to the whole. *250'65. F: P e f. a e sect'P- 'P- P. 3 C a. D..~ P smor P /3} Denm. F. 40-16. F: Hp.. p'P"C'(P a)C pP"C'(P) (1) F.*211-133. F: Hp. a~e. el..a= C'(Pa). [*211'703]. a!p'P"C'(P a). [(1)] D. 3! p'P"C'(P C) (2) F. (2). *40-6-62. D F: Hp. a e 1. a! P. D.! C'P n p'P"C'(P /3). [*208'47].- {P smor (P C 3)} (3) F. *2111. *24-13.) F: P= A.. sect'P- G'C'P = A (4) F.(4). Transp.: Hp..! P (5) F. *200'35. *250-104. D F: Hp.! P. a e 1.. {P smor (P,8)} (6) F. (3). (5). (6). F. Prop *250-651. F: P e f.. Nr'P n P C"(sect'P - LC'P) = A [*250-65]

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 *250-652. F:PEBord. QCP. a!G'Prp'P"C'Q. (P smor Q) [*208-47] *250-653. F:PeBord.H! C'Pnp'P6"(nCu 'P).).r-,'(PsmorP~a) Dern. F. *37'41. D I. J'(P C a) C a n G'P. [*40-16])F*p'P"@An ('P)C Cp'P"G'(P t a) (1) [*250-652]). {P fsmor (P C a)}: F. Prop *250-66. F: P E a e sectP. P smnoi (P C a).).ct= C'P [*25065. Transp] *250-67. F: Pefl xC'PP.).cfPsmor(P lP',,x)} Dem. F.*211-302.DF:Hp.).Pxe sect"P (1) F.*200-

52.)F:Hp.).P'x:jCGP (2) F.(1).(2). *25065.) t. Prop *250-7. F:. Pefn.:xe CG'P.:,. P P*'xe f2: P e Ser Dem. F. *250-141.) F:. P e f2.): xE C'P.)x. P P*'XE 7 f2 (1) F. *2500121.)D F:. X (J'P. DX. P C P*'x e f2:_: XE C'P.! a A n C'(P P*'x). DX,,, E! min(P C P*c'x)'a: [*202-55])D: x e UI'P r a..:)x,. EE! min (P C P*'x)'a: [*205-27]),,a. E! minp'a: [*10-23]):a!(6PP a.),.E!nminp'a (2) -4 F.*205-18. *202-52.) F~: Pe Ser. a= B'P. D.E!itninp'cx (3) F. (2). (3). D:. x e CIP - DX. P ~ P*~xxe f2: Pe Ser: D a! a A CGP.,,. E! minp'a: [*2500121] D:Pef2 (4) F-.(1). (4).)D F. Prop This proposition is used in proving that the series of ordinals in order of magnitude is well-ordered (*256-3). We prove first that if PE f2, the ordinals up to and including Nr'P are well-ordered; thence, by the above proposition, it follows that the whole series of ordinals is well-ordered. R.& W. III. Ici

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*251. ORDINAL NUMBERS. Summary of *251. The name "ordinal numbers" is commonly confined to the relationnumbers of well-ordered series, and will be so confined in what follows. The relation-numbers of series in general are commonly called "order-types*." Thus a is an order-type if a e Nr"Ser, and a is an ordinal number if a e Nr"f. In the present number we shall be concerned with a few of the simpler properties of ordinal numbers and of the sums, products, and powers of wellordered series. We put NO = Nr"f Df, where " NO " stands for " ordinal number." We prove in this number that any relation similar to a well-ordered relation is well-ordered (*251'11), and therefore any relation similar to a well-ordered series is a well-ordered series (*251'111). We prove *251'132-142. F a e NO.. a+ i eNO.. i - +aeNO *251'1516. F. r,, 2r e NO *251'24. F:a,/ eNO. D.a+-/ eNO We prove that if P is a well-ordered series of mutually exclusive wellordered series,,'P is a well-ordered series (*251'21); that if P is a wellordered series of series, I1P is a series (*251P3); that if P is a series and Q is a well-ordered series, PQ and P exp Q are series (*251'42); that if P, Q are well-ordered series, so is P x Q (*251*55), and therefore the product of two ordinal numbers is an ordinal number (*251'56). In virtue of the uniqueness of the correlator of two well-ordered series, we have *251'61. F:. P, Q e Rel2 excl. C'P C C.: a! (P smor Q) n Rl'smor.. P smor smor Q whence, without assuming the multiplicative axiom, * We shall also speak of them as "serial numbers."

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SECTION D]] ORDINAL NUMBERS 19 *251'621. F G'P C fl. E [(P sSm-or Q) Q RIsinor.). YNr'P = INr'Q. 7INr'P = fl Nr'Q *251-65. F:aeNO-"IA./3ENR.Pe/3. C'PCa.). Nr'P =f * a. flNr'P = a ex p,. 3 Finally, we have propositions (*251P7- 71) showing that the existence of an existent I2 in any type is equivalent to the existence of 2, in that type, and therefore holds for every type of homogeneous relations, except (possibly, so far as our primitive propositions can show) in. the type of relations of individuals to individuals. *251-01. NO = Nr"fn Df *25-1. F: a E NO.. (HP). Pe f2. a=Nr'P [(*2511)1)] *25-11. F:PEBord.PsmorQ. D.QEBord

Dem. F. *2058. *250-1. *37-431.) F:. PE Bord. SEP smor Q.): a C C'P. a! a.),. f! minQ'S"aC: [*3763431] D: 8e S"CCCI ex'C'P. g! 8. Do. a! ininQ/: [*71-491] D: /3E Cl ex'S"'C'P. DP. a! miuQI'/: [*1 51K11-131.*37h25] D: / E Cl ex'G'Q. Dp. q I mi I IQ': [*25001] D: QEBord:.DF. Prop *251-111. F:PEI2.PsmorQ.D.Qe!2 [*251-11.*204-21] *251-12. F: PE Bord. D. Nr'P C Bord [*251-11] *251'121. F: Pe f2. D. Nr'P C f2 [*251-111] *251-122. F: aENO. D. aC f [*251-121-1] *251-13. F: Pe Bord. zr, e C'P. P.P- z E Bord Dem. F. *205-83. *250-1. D F: Hp. a! CG'P ra. D.a! min (P —+z)'a (1) F. *205-831.)F:Hp.C'(P-j+z)a==fz.).a!rninn(P-+z)'a (2) F.- *161-14. D F:. Hp. [! C'(PS, z) n a D: a! C'P A a *v. C'P n a = A.! t'z n a: [*161-14]!:! C'P n aa.v.' C(P-f+z) A a = t'z (3) F:. Hp.): a:! C'(P-i+z) A a.,. a! min (P-f+ z)'a (4) F. (4). *250-101. DF: Pe Bord. zrEC'P.'. P +>ze Bord (5). *250014*104. *20041. D F: P.I ze Bord. D, Pe Bord.zhe' CIP (6) F - -().).Prop

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20 SERIES [PART V *251-131. F:Pcfl.z-reC'P.=-.P+>zegl [*20451'.*251P13] *251-132. F:aeNO.=E.a-j-ENO Dem. [*181-11.(*181V01).*251-131]. P+> XE n [*181P3.*251-1].Nr'P i- 1 NO F-.(1). *251-1.) F-.Prop *251-14. F:PeBord.z~cC'P. =E.z+j-PcBord Derm. IF. *205'832. *16112l.)d- F-:. Hp.): z e- E a.).min(z4+-P)'a =minp'la: [*25101]:a(an 'P). Z(E~D. a! min(z<4- P)'a F-. *205-833. *161-12.)D I- Hp.zcEa. f! P.).a! min(z<+1-P)'a F:. Hp. f! P.)D::a! an O'(z+IP).D)a a! mmn (z4~-P)'a: [*250-101]):z*fPcBord F.-*1 61'201.*250-4. D)F:P=A.)D.z+FPeBord F. (3). (4).F: P eBord. z e G'P.). z -Pe Bord F.*250-14-104. *200A41.)F:z +FP Bord.D. Pe Bord.ze'-.. G'CP F.(5). (6).)DIF. Prop *251-141. F:Pef2.z~cC'P.=E.z*-JPeI2 [*204-51. *251-14] *251-142. F:aeNO. -.i1+aeNO [Proof as in *251-132] *251-15. F. 0,,ENO [*250-4. *153-11] *251-16. F. 2,.E NO [*250-41. *153'211] *251-17. F: x+ y. xtz.,y z.)D. xJ +Iz cf2 [*251-131. *250-41] *251-171. F. 2,.- i-e NO [*251-16-132] *251-2. F: PeC ReP 1excl n Bord. G'P C Bord Y.).'P E Bord (1) (1) (2) (:3) (4).(5) (6) I Dem. F. *162-2-3. CFIaYr4 '7P. D.a2! a nF"G'CIP. [*37-264] D. a! C'P n Flla F. *37'46. *33-5.:) F: Q eF61a. D. a! a n C'IQ 1.(1). (2). *250-101.)D F:. Hp.)D: a! a n C'I'P. D. (HQ). Q minp Fla. a.' minQ'cz. [*205-85]). a! min (I'P)'a F. (:3). *250-101.:)F. Prop (1) (2) (3)

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SECTION D] ORDINAL NUMBERS 21 *251-21. F:P E Rel12 exci A f2. G'P C ft.) Vcp E fl [*204502.*251-2] *251-211. I-: Nr'P e NO. Nr"C'(P C NO. '. Nr'P E NO Dem. F.*18216162.)F:lllpJ). Nr'4.;PeNO.4.PcRelexcl (1) F.*182-05-11. *151P65.:) I-: Hp.:). Nr"C' J.;P C NO (2) F.(1). (2). *251,122.:) F: Hp.);p E Rel12 exci A f2. a';p c f2. [*251-1.(*183-01)]).~Nr'P eNO:: F. Prop *251-22. F:P,QeBord.CG'Pr G'Q=A.).Pt-QeBord Dem. P 4. Q E Bord.- Cf'(P 4. Q) C Bord. P 4. Q e Rel2 exci. [*251-2]). P-Q E Bord(1 F.-*160-21. *250-4.) F: P~.Q=A)Pt-Q eBord (2) F.- (1). (2)..)F. Prop *251-23. F:P,Qef2. G'PnC`Q=A. ).Pt-Qd2 [*204-5.*2951P22] *251-24. F:a,/3cNO.)c-i-fl~eNO Denm. F. *251-111.*180-1211.:) F: P, QE!~ ).4. (A A C`); t;P E.I (A A C1'P) 4.; t; Qe n. Cc4. (AA G'CQ);t;PAn C'(AAn C'P)I; t;Q = A.

[*251'23.(*181L01)]).P + Q E!72. [*180-3.*251-1]).Nr'P -i-Nr'Q e NO(1 F.-(1). *251-1.)F..Prop *251-25. F: PtQeQ.=.P, QEfil GPA C'Q=A Dem. F. *204-5.) F: Pt-Qe fL D. P, Qe Ser. G'P n C'Q =A(1 F.(1).*205-84.:)F:.Pt-Qef~.):a!C'Pnc.a.Da! minp'ct: [*250-11]): PeBord (2) F.-(1).-*205 841.)F:. P-t-Qe6f2.)D at I a - C'lP A C'(P t Q).),, MD! miQ'(a - CP

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22 SERIE9.[PART V.[*160-14.(l)]): a! a rn C'Q. Da 2! MiDQ'(a- C'P). [*250-101]) Q eBord (3) F.(1).(2).(3).) F-:Pt-QEf~. D.P,Qef2.C'PnG'"Q= A (4) F.(4). *251-23.) D F. Prop *251-26. -: a, flENO -tiA.=. a-i-/3 NO -L'A [*251-25] *251-3. F: P e f. C'PC Sere.D. ll'1P cSer [*204-57. *250'1] *251-31. F:E!!B"lC'P..B['C'PcF&'lC'P Dem. F. *71-15 71.DF: Hp,. D. B rGPe61 -+ Cls. (B rC'P)=U'P (1) F.*93-103.)D F. B C- F (2) F (1).(2). *80-14.)DF. Prop *251-32. F: E!!B"IC'P.ft!P.).BrC'P=B'll'P Demn. F.*1 72-16 2.DF:Hp.)D..B'H'P = B4P'P [*82-21] = tl'(B r C'P): D F. Prop *251-33. F:C'PCfl-t,'A.fj!P.).f!ll'P.BrC'P=B'FJ'P [*250-13. *251-32]. *251-34. F: P e Re 1 exci. C'P C fl - t'A.) D!c e4P"U'P Dem. F.-*251-33. *173-16. F::Hp. ft!P.D. f! Prod'P. [*173-161]). a! Prod'C"GIP. [*115-1]).a2!E4'jC"G'(P(1 F. *83-15.DF: P= A.)D.a2!e4'C"GC'P (2) F. (1).(2).)D F..Prop *251-35. F:: Pdci.):D aPC1f3. a, /3 e C1'C'P: (a[z). z E a - 18 a A PIZ =8 /3 P'z Demn. F. *170-2.)D F:. a,/3ecCI'G'P: (az). Z ca -/3.a nPI 'z=/3 PIZ:D. aP,1/3(1 F. *170-23-1. *250-121.)D F::llp.:. aP~18.):a,/3eCl,'C'P:(a~z).zea-13.'ae-P'z=/3AP'z (2). (1).(2).)F~. Prop

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SECTION D] ORDINAL NUMBERS.if9.3 4- 4 a,18ECI'C'P:(H[z).ze/3-~a-anP'z=/3AP'z [*25P135.*170-101] *251-36. F:PefL-).P.lESer Dem. (1z, w). z ea -/3. w 6/3-7. a~ A P'z =13nP'z. /3A P'w =y n P'w (2) F. *201 14.) F:. lip. zc a c-J3. w E/3 - ry. a~ A P'z =/3 A P'lz. /3 A P'w = y A Pi.D ZPW. D.-ZE a-'7. aA P'Z=, Y nP'Z (:3) F. *201l14. DF:.llp (3).D: wPz.D. w a - y. a nP'w rA Pw (4) F. *250-121.) F: Hp. a~, 03 E C'ICI'P. a #/3. D. ({z). z = rnrnp'{f(a - 13) v (i3 - I [*251'35] D. a~ (Pl1 w: Pe1) la (6) F. (1).(5). (6). D F. Prop *251-361. F:PEfl.). Pl~eSer [*251-36.*170-101] *251-37. F: P'6f~:) *P=Pdf [*251-35.*171V2] *251-371. F: Pe6il. D. Pl~= Pf(I *251-4. F:P eRel13 arithm A Bord. CU'P C Bord. C'V'P C Bord.). eBord Dem. F.*2 51 2.)D: Hp. Y., TPe Rel12 exci A Bord. C'$'PI C Bord. [*251-2] D Y.,'YP e Bord: D F. Prop *251-41. F:P e~rtun2.'~lCYP f2:.,:',f [*204'54. *251V4] *251-42. F:PESer. Qef2.).PQ,(PexpQ),ESer [*204-59.*250-1] *251-43. F:aENR.aCSer.183ENO.).(cexp.18),ENR.(aexprfl)CSer [*186-13. *251-42] *251'44. F:a e NO - tf0O.. 3EcNO -t'01. a.cexpr13t+0r Dem. F.*165-27.) [*251P33.*176-1]) A! (P exp Q)(1 F. (1). *186-13. D F. Prop

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24 SERIES [PART V *251'5. F:fj! P. QeBord.).Pj.;QeBord [*165-25.*251-11] *251-

51. F:ft!P.Qef2.).PjAQEf. [*165-25.-*204-21.-*251P5] *251-52. F:Pc Bord.). C'P 4.Q CBord [*1 65-26. *251-12] *251-53. F:Pdf2.:).C'Pj,;Q4~ [*165-26. *204-22. *251-52] *251-54. F:P, QEBord. D.P xQc Bord Dem. F.*165-21. *251-552.) F:Hp. f[! Q. D. Q 4.;P E Rel2 excl n Bord. C'Q;P C Bord. [*251P2.*166-1] D. P x Q c Bord (1 F.-*166-13. *250-4) F: Q =A.:). Px QE Bord (2) F. (1). (2). DF. Prop *251-55. F:P, Q 6f. D. P xQE f2 [*251-54. *204-55] *251-56. F ac, 3eNO.). a *,3e NO [*184'13.*251-55-1] *251-6. F: P, QERe12 excl. C'P Ct f. ScPsgmor Q ri RI'"smor. = 1(21N). Nc Of'Q. X = (S'(N)! -iio- N}.D t cu c6 661k. ktfrzc P smr-or s-mor Q Demn. [*83-43] D. L /hEE'j ". (1) [*164-43] D. cit6 fP P smor smor Q (2) F. (1). (2).)D F. Prop *251-61. F:.P, Q cRe 12excl.G'PC f2.:! (P smor Q) nRl'smor.. Psrnor smor Q Dem. F.*251-6.)F:Hp.g!(Pghi~iioQ)nRl'smor.).PsmorsmnorQ (1) F.(1). *164-17.) F.-Prop *251-62. F: Hp*251V61. [!Psrnaor QrnR~lsmor.). IT' smor Y.'Q. HIP smor H'1Q. ~Nr'P= Nr'Q. f Nr'P = H Nr'Q Dern. F. *164-151. *251-61.) F: Hp.). I./Psmuor Y',Q (1) F. *172A44.*251P61.) F:Hp.). fl'lPsmorIII'Q (2) F. (2). *185-1. D)F: Hp.).llHNr'P=flINr'Q (4) F. (1). (2). (3). (4).:)F.Prop

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SECTION D] ORDINAL NUMBERS 25 In the above proposition, the hypothesis " P, Q e Re 1 excli" is unnecessary fior I Nr'P = Y. Nr'Q and II Nr'P = II Nr'Q, as appears from *1 83-11 and *185-11. Thus we have *251-621. F: C'P C fI. P! (P smor Q) rn Rl'smor.) 'SNrP = Y.,Nr' Q.I NrlP = llNr' [*25)1-61. *183-11. *185-11] *251-63. F:aeNO-t'lA./3EsNR.PeRe 12excl.Pe/3.C'PCa.:). Z'6PeI3;xa. Y.,Nr'P=13a Dem. F-. *164-47. *165-27-21.)D F:Hp. Q E a *at Or.. QI;P E8. C'IQ 4,;PCa. P,94,;P ERePl excli. [*164-47]:. H!(Q,;P) sinor P AR1'smor.P, Q,;P eRel exci. [*251P61]).(Q 4,;P) smor smor P. [*164-151.*166-1]) (P x Q) s nor! "'P. F. *162-42. Transp.)D F: HP -a =Or.) "PA [*184-16]).Z'P ~~a (3) F.-(2).-(3). D F: Hip-D.)Y., 6'Pe/3*a (4) [*183-13]:). I Nr'P =,83 a (5) F. (4). (5). D F. Prop *251-64. F:Hp *251-63.).HllP E(a expr j3). INr'P = aexpr o [Proof as in *251L63] *251-65. F:aENO -t'A. 3ENR.PE/38.CGPCa.). I Nr'P =/3;a. [lHNr'P a expi3 Dern. F. *182-16. *183-231. F: HP.QEa.:). 4,;PeRel12 eXC]. 4,'Pe Nr'lP.G';P CNr'Q. (1) [*251P63] Y., ~Nr' 4;P = Nr'P *~ Nr'Q. [*183-14]. ~Nr'P = Nr'P *< Nr'Q [*152-45] =fl"~<a (2) F.(2).*1O-23.:)F:Hp.:).INr'P=/3*(a (3) F.(1). *251P64. D F: Hp. Q E a.). JNr' 4,P - (Nr'Q) expr (Nr'P). [*185-1-12]) IlNr'P = (Nr,'Q) exp, (Nr'P) [*152-45] = aexprl (4) F.(4).*1O-23.)F:Hp.).l-lNr'P=aexpi3 (5) F. (3). (5).:) F. Prop

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26 SERIES [PART V In virtue of the above proposition, the usual relations of addition to multiplication, and of multiplication to exponentiation, when the sumnands or the factors are all equal, can be established without the multiplicative axiom, provided the summands, or the factors, are ordinal numbers. *251-7. F:if - tA n t0'Ia..! 2,t'c2 2 n ta'c. = 2, Dem. F. 64-553. D t::HE i - t'Anton,,'a.= -.(HEP)~ P e - L'i. c'Pc t,'a (1) F. *200-12.)F P e fI - t'A.). (ax, y).

x, yE CJP. x + y. [*153-201.*553]. a! 2, A Rl'P (2) F. (1). (2).) F: a fl - D 0'.). (3P). C'P C t0'ct. a! 2,.r A Rl'P. [*33-265]). (HQ). 9 E 2r. C'Q C to'a. [*64-55] D. j! 2, A t.'a (3) F. *251K16K122. DF:a! 2, 'at0'a.D.!fl - 'At n 0' (4) F.(3).(4). F fli - t1A n t,)01,. a!.E 2r n too'a(15 F-. *64~55. F ' 2,. n to,'ca. (ax,~ y). x + y. x~ X, y e t,,Ia. [*63-62] -C.(x, y).xry. ta [*54-26].! 2 A t'a (6) F.(5).(6).(*65 01) F. Prop *251-71. F.E!!f - tfA A t,,'Cls.! f - tAn t,,'Rel [*251-7. *101-42-43]

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*252. SEGMENTS OF WELL-ORDERED SERIES. Summary of *252. The properties of sections and segments are greatly simplified in the case of series which are well-ordered, owing to the fact that every proper section has a sequent, whence it follows that the class of proper sections is P"C'P; and this is also the class of proper segments. Hence also the series of proper sections or of proper segments is the series P;P (*252-37). The series of all sections is P;PC'P (*252-38); hence (*252-381) Nr's'P* = Nr'P +- i. The most useful propositions in this number are (apart from the above) *252-12. F P e.. sect'P - t'C'P = D'Pe - t'CP =P"C'P. sect'P = P"C'P v t'C'P *252'17. F: P e Q - t'A.. sect'P - 'A = P"('P v 'C'P *252-171.: P e f. D. sect'P - ' 'A - 'P P= iP *252-372. F:. P e f.: s'P e: E! B'P.).Nr'C'P = Nr'P: E! B'P. D. Nr's'P = Nr'P + i *252-4. F: P e. X sect'P. [!. 3. p' e X *252'1.: P e I. a e sect'P - 'C'P.. E! seqp'a [*250'124] *252'11. F: P e f... sect'P - 'C'P = sect'P n ('seqp Dem. F. *206-18-2 F. C'P - e I'seqp (1) F. (1). *252'1...Prop

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~28 SER~IES [PART V *252-12. F: Pe.L). sect'P - tfCGP = D'PE - tIC6P = P"C'P sectfP = P"CC'P v VCI'P Dem. F.*211L24.*252-11.)F:Hp'.xe sect'P-tfC'P.).aED'PE (1) F. *211 15. D F: Hp.a e DIP, -- fG'P. D. a. E sect'P - tcC'P (2) F. (1). (2). F: Hp.:. sect"P - tIOPP = DIP, - tICIP (3) F.*211P302.*252-11. F F: Hp.)D. sect'P - fC' = P"P" P (4) F.(3).(4). *211 26. D F.Prop In dealing with sections and segments of well-ordered series, it is necessary to distinguish series with a last term from such as have no last term. If a series has no last term, C'P = P'P'P, so that G'Pe D'Pe. But if a series has a last term, C'P e D'Pe; in this case, D'P = P1"C'P. Thus D'Pe is either PI"G'P or sect'P, according as there is or is not a last term. In either case, sect'P = P"G'CP V t'C'P, as has been already proved in *252-12. *25213. F:P fI. E!B'P.). sect'P-tfCP=D'Pe=P6"C'P. sect'P = DIP, v t(CC = P1"G'P v tf G'P Demn. F.- *250-21. *211-36.:) F: Hp.).sect'P - D'PE = t'C'P. [*24-492.*211K15]) sect'P - tf 'P= D'P([*252-12] -P"'CP (2) F. (1). (2). *21126. D F. Prop *252-14. F:Pef2.,E! B'P.). sect'P= D'P =P"C'P v tfG'P [*250-21. *211361. *252-12] *252415. F: PeI2.). D'PD =P"D'P v ifD'P Dem. F.*252-13. D F: Hp. E!B 'P.).D'PC=P"1(D6P uL'PPB'P [*202-524] = P",D'P w tfD'P (1) F.*252 14.):FHp. E! B'P..D'PF = P"D'P v t'D'P (2) F. (1). (2). D F. Prop *252-16. F: Pe f - 2,..DP, = sect'(P C D'P) Dem. F.*204-271. D F: Hp. D.D'Pr-i 1. [*202-55]). G'(P C D'P) = D'P. [*250-141.*252-12]) sect'(P C DIP) = P I D6P"D'P v tfD'P [*37A42-421] - P"D'P v ifD'P [*25-2-15] = D'Pe:) F. Prop

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SECTION D] SECTION D] SEGMENTS OF WELL-ORDERED SERIES 2 29 *252-17. F:P E - t'A.. sect'P - -AP"U'IP vtr Derm. -. *252-12.) IHp.:). sectrP - '= (P"rC'P-t~'A) vt LP' [*3341] = ~~~~~~P'c(cl'] v t`C'P::) F. Prop *252-171. F:P eQ.). sect'P- L'A - t'CP =Prc Dem. F. *252-12.. F: Hp.). (sect'P - t'C'P) - t'A =P'W'~P - A [*33-41] = P"(I'P:) F.Prop *252-3. F:1Pefl.).D'rS'P*=P"cC'P [*212-171.*252-12] *252-31. F: Pel.~ f! P. D. C'rsP* = PcG'P v fcc'P [*212-172. *252-12] *252-311. F:Pef2.a!P. D. U'63'P*= P"E[cP v cl [*212-171. *252-17] *252-32. F: PEQ f.:). D'S 'P = P"ID'I [*212-132. *252-15] *252-33. F::P e f2- D.. CIS'P = P"cD'P v t (D'P [*212-133. *252-15] *252-34. F: PeSI2. E! B'P. D.C'S (P =P"GCP Demn. F. *202-524. D F: Hp.:). P'B'P =1)'P. [*252-33] D. C's'P = P"GC'P::) F. Prop *252-35. F: P c f2-t.E! BIT. C'S'P = P"GcP v t'C' [*212-133. *252-14] *252-36. F: P cf2. E! B'P.SIP =P;P Dern. F.-*212-25. *252-34.)D F: Hlp.:).P;P= (SIP) ~(C'('P) [*36-33] =-S'P: DF. Prop *252-37. F:Pcf2.).(s'P4~(-tIC'P=P;P Dem. F. *36-3. D F. (S P) I (tC'P) = (s IP) (0G'IP - ffG' CP) [*212-133-134] = (SIP) (D'PI~ - t'G'P) (1) F. (1).*2052-12.) F: Hp.D.(SIP) (t'C'P),(S'P.) ~(P"cC'P) [*212-25] =PP:) F. Prop

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30 30 ~~~~~~~SERIES[PRV [PART V %a ~-+ *252-371. F:Pefl.rE!B'P.D).;P=P; P-i-WCP Dem. F. *212-25. *252-32.F. *212-133. F.*252-32. [*200-12.*2043S4] F -(1). (2) -(3). *204-461 F-. *212-134. *161V2. F. (4). (5).)D F. Prop D F: Hp.:). P; P = (S'P) DsP D F: Hp. t! P.D. C'P= B'Cnv'Is'P D. DISTP II)F: Hp. A! P.:). P;P-+G~P =SI D F: Hp. P =A.. SIP-=A. P;P+FWP=A (1) (2) (3) (4) (5) *252-372. F:. Pef~.D:sl'1lEf1:E!B'1-).).Nr'g'P=Nlr'P: Dem'. F.*252-36.*204-35.)F: Hp.E!B'P.D). c'PsmorP. [*251K111.*152-321] D. s,'P E f. Nr'I'P Nr'IP (1) F. *252-371. *204-35. *20052 2.)D F: Hp. E! B'P.D. Nr'IS'P =Nr'P -.(2) [*251-132] D. 3'Pe E2 (3) F. (1). (2). (3). D F.'Prop *252-38. F:PcE.). cP = P;P+>'P Demn. F. *252-12. *212-24. D [*37-6.*200-52] (ax, y). x, y e C'P:. a P'x. fi P'y P'x C P'y. P'fx f P'y. V. -4l (axw). x E'P. a = P'X.f =C'P: [*204-33 34] =: (ax, y). xPy. a = P'x. / PY. V. (ax).xc G'P - a=PX, tc [*150-522] E:a(Pli)/3..v.ae C'pP.fl0'cp: [*161-11]:a(P;P -f*GP)/3:: D F.'Prop *252-381. F::Pe fl). 'P*,e f. Nr's,'P* =Nr'Pt [252-38. *200-52. *204-35. *251-131]

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SECTION D] 31 SECTION D] SEGMENTS OF' WELL-ORDERED SERIES 3 *252-4. F::Pefl.XC sect'P. 21! X.).p'XEX Demn. F. *211-441.:) F: HP. P= A.). X =tiA. [*53-01] D. p'xex (1) [*252-381.*250-121] D. E! min ('&'P*)'X. [*210-222.*211-67-66]). 'x E X (2) F. (1). (2).) F. Pi-op *252-41. F:Pcfl.XCsect'PKL!\.).s'XcX [Proofas in *252-4] *252-42. F:. P ei. (Cnv','P*)i"o- Coa-: X C a-. a! X A C'1,'P*. DX. s'I(x r C' 13'P*)C E a-:. (Cnv's'P*)"ao- C a[*20'0361. *252-381. *212-322]

*252-43..F: i.('*1a - X C a- X a! X s G'IP*.)A. p'(X A C'S'P*) E a-:). (S'P*)" a- C aDem. F. *212 181. D F. (Cnv',g'P*) smor (,g'P*)(1 F. (1). *252-381. DF: Hp.D. Cnv','P* e f (2) F. (2). *212-34. *250-362.) F. Prop

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*253. SECTIONAL RELATIONS OF WELL-ORDERED SERIES. Summary of *253. In the present number we shall consider the properties of the relation Ps (defined in *213) when P e f. The relation Ps has great importance in this case, owing to the fact (to be proved later) that Nr"D'Ps is the class of all ordinals less than Nr'P, and that, if P, Q are any two well-ordered series, either P is similar to a member of C'Qs, or Q is similar to a member of C'PT, whence it follows that of any two unequal ordinals one must be the greater. The present number consists merely of the more elementary properties of Ps when Pe l. The interesting properties connected with greater and less will be treated in the following number. The most useful propositions of the present number are the following: *253-13. F: P e f. D. D'Ps = P C P"(I P = P "P"C'P *253'18.: P e 2.. C'Ps C P C PC"'P v t'P. C'Ps C f Instead of C'Ps C P C "P"(I'P v t'P we shall have equality, unless P=A (*253-15). *253-2.: P e 2 - 2,.). Nr'Ps = Nr'(P U ('P) + i The case when P e 2r has to be excluded, because then P (P'P = A. *253-21.: P e f.). i + Nr'Ps = Nr'P + i This proposition involves Nr'Ps = Nr'P when P is finite, but when P is infinite it involves Nr'Ps = Nr'P + i (cf. *261'38). *253'22. F: P e f. 3. Ps D'Ps smor P (1'P *253-24.: Pe.). Ps e l *253'4.: Pe - fA.. C'Ps = Q {(R). P = Q - R. v. (ax). P = Q+x} *253'421. F: P e Q. Q D'Ps.).(QsmorP) *25344. F:a,/3eNO - A.8 r0,...a+,ia+a

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SECTION D] SECTIONAL RELATIONS OF WELL-ORDERED SERIES 33 This proposition marks a difference between ordinals and cardinals. An ordinal is always increased by the addition of anything at the end, whereas this is (often if not always) not the case with a cardinal if it is reflexive and greater than the addendum. The above proposition ceases to be true if we add /3 at the beginning instead of the end: /3 - a = a will be true if a is infinite and /3< co is not greater than a. (For the definition of o, cf. *263.) *253-45. F:aeNO - t'A - Or.. a - i a Similar remarks apply to this proposition as to *253'44. *253'46. F:Pe.Q,Re 'P,. QsmorR.D.Q=R I.e. no two different sections of a well-ordered series are similar. It follows from *253'46 that the series of the ordinals of proper sections of a well-ordered series P is similar to the series of proper sections, and therefore, by *253'22, to the series P with its first term omitted (*253'463). We have next a set of propositions (*253-5 —'574) on the circumstances under which Nr'Ps = Nr'P and those under which Nr'Ps = Nr'P - i. As a matter of fact, the former holds when P is finite, the latter when P is infinite. But the distinction of finite and infinite will not be introduced till the next section. In the present number, we prove that (assuming Pe fI) Nr'P, = Nr'P if ('PP = (CP. E! B'P, and if not, then Nr'P = Nr'P + i (*253'56). This is proved by using PI as a correlator. (PI as a correlator moves every term one place down, except the first, which disappears.) For, if Pef, we have Pi;P=P D'P(*253'5); hence we prove PC('P, smorP D'P (*253-502), and hence, if ('Pl-= (P, we obtain P (I'PsmorP C D'P (*253.503). Hence by *253'2 (with special consideration of the case when P e 2r) we have the two propositions *253'51.: P e f. (I'P = (I'P. E! B'P. B. Nr'Ps = Nr'P *253'511. F: Pe f. 'P = ('P. E! B'P. D. Nr'Ps = Nr'P + i. Nr'P ('P = Nr'P But if there is a term, say x, belonging to ('P - (I'P, use PI as a correlator for the predecessors of x; we thus find that, in this case, P smor P CIP. Hence, by *253-2, Nr'Ps = Nr'P + i. The hypothesis ('P, = (I'P. E! B'P means that there is a last term, and every other term has an immediate successor. This, as we shall prove later, and as is indeed obvious, is equivalent to the assumption that P is finite but not null. From the above propositions it results immediately that *253-573. F:. P e n.:: (P, = ('P. E! B'P. E. i - Nr'P + Nr'P Hence it will follow that finite ordinals other than Or are those which are increased by the addition of i at the beginning. We have also R.&W. III 3

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34 SERIES [PART V *253-574. F:.PE~I-t'A.):(IPP=cI'P.E!B(P.=-.ijNr,'P=Nr'Pj-i Whence it will follow that finite ordinals are those for which the addition of ii is commutative. *253'1. F:. PEQf. D: QPfR.. Dem. (aa,/3).-a8,I3P"UIP v tC'IP.!13-a. Q=P aR = P /3 F. *213-1. *252-17.. F:. Hp. ft! P.): QPsR. F. *33-241.) F:. P = PA.: P"P v 'G'P = tfA: (aa,13).a,3EP"("P~vt'C'P.a!38-a.Q=P~a.R=P~13 (2) F.(I).(2).) F. Prop (ax, Y). XE PCP. aXPy. Q = P ~ P'x. 1?R P ~ P'y. V. Dem.(a) '.Q=P PXR=P F. *33-152.) F: a=G'IP.13e6P"G'P U t'C'P.)..!13- a (1) F. *200-52. (1).)F: Hp. a eP"PP.1 = C'P.:). H!1 -a (2) (Hja, 13.a,8 13 P"PI~P. j!8 1- a. Q =P ~ a.]? = P 1.v. *376.*3-33](aa,R1). a E P,11"PP.13= C'P. Q = P ~ a.]? = P ~ 13: (ax, Y). X, y E PIP. a! P"'y - P'x. Q = P ~ PCX. R = P ~ P"'y. V. (ax). w E P11P. Q =P ~ P'x. R = P: [*211461.*210-1] (ax). x e PUP. Q = P ~ P'x. RB P: [*204-33'34]~(axe,y). x, y ePP. xPy. Q = P ~P'x. R =P P'y. v (ax)..x ePCP. Q =P P'x. R =P (3) F.(3). *33-14. D F. Prop *253-12. F:PEf2.Pe-e2,.).Pg=(P~i;P~p1'CP)+f*P Dem. F. *2O4-272.) F: Hp.).P[Ps1..l [*2O2-55.*213-151]).P V"P"'(jPP= (J'P;P;P ~ PIP (1)

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SECTION D] SECTIONAL RELATIONS OF' WELL-ORDERED SERIES 3 35 F. (1). *253-11.. F:: Hp.):. QPsB - Q (P; p; P G'11P) B. V. Q C'(P ~;ii;p U',P).1 = [*161-11] =Q{(P~;P;P~(U'P)-I~P}R::)F.Prop *253-121. F:PEf2.).Pr —.EC'Pt;P;P~%'P Dem. F. *200O52.) F: Hp.)CTPr P"U'ip. [*36-25] D. P E'ptP; PV I'P: DIF. Prop *253-13. F-: P efl.:).Dis= 6666 V 6 CI Dem. F. *213-141. *252-171.)F: Hp.:). D'P = P ~"P"WjP (1) F. *37-22. *2-50-13. D F:Hp4ft!P.).P~,P"~C'P = P V"P"'P v t'lP ~P6B'P [*33-4l.Transp] = P "`i'iGP v." (2) F. *250O42. DF: Hp. f! P.). AcP ~ 6P"U'1P (3) F*33-241.) DF: P=A.).Pt"c~P"G= A. Pt"cp(P"PPA (5) F.(4).(5).)F: Hp.:).P~I"P"cC'P=P~6"P"U~P (6) F. (1). (6). D F. Prop *253-14. F: P ef2. D (lips = (P ~"iipl'u'Pv t'P) - i/A = (P ~cPlcC'P t'P) - tIA Dem. F. *213-162.)D F: Hp.D.G (i'P5= P ~"Isect'P - tA [*252-12.*36-33] = (P ~"P"'TP v i/P) - t/A (1) [*253-13] = (P ~r"P"E1'~P v VIP) - t/A (2) F.(1).(2). FProp *253'15. F:P cf2 - t/A.)C'Ps =P P"Pv~cpcc jt'P =P PCP c v tip [*253-13'14] *253-16. F:Pe6fl-tiA.).B'~= A.B'Ps=P [*213-155'158.*250-13] *253417. F:PE!1.:).PsD,'Ps=P~;P~PVI'P Dem. F. *253-11.)D F:: Hp.)D:. QPgR. =: Q (Pt~;P;P ~PIP) R. v. Q eP V"Pt P. R= P:. 0 ~~~~~~~3-2

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36 SERIES [PART V *253-18. -: PEQl.).C'PsCP~""P"PClPv ffP.G'lPsCf Demr. 1-. *253 11.) F:: Hp.)D:. QEU'CPs. D: ([x)..xcP(IP. Q = P ~P'1x. v. Q=P: F..(1). *250-141.) F: Hp.:). C'Ps C f2 (2) F.(1).(2).)IF-.Prop *253-181. F: P eI2.:). C'Ps C D'Ps v t/P [*253-18-13] *253-2. F:P e n- 2r.)Nr'P = Nr'r(P ~'P)4- iDem. [*213-151.*252-171] = Nr'P P ~ (I'Pi I [*204-34] = N r'(P ~ (I'P) -1-1: F. Prop *253-21. F:Pef2.D.i-jNr'1Ps=Nr'1P-j-i Dern. [*204-46-272] =Nr'Pi- (1) F. *213-32.:) F: P e2r.). Nr'Ps I2, [*161-211] =2r4Ki [HP] = Nr'P i- (2) F. (1). (2). DF. Prop It would be an error to infer from the above proposition that Nr'Ps = Nr'P, since addition of ordinals is not in general commutative. When P e ~il, Nr'Ps = Nr'P holds when (frP is finite, but not otherwise. When C'P is not finite, 1 —Nr'Ps = Nr'Ps, so that Nr'Ps = Nr'P -1;but Nr'1Pt Nr'P +i-. *253-22. F:Pe Q.). Pst~D'Pssmor P ~U'1P [*253-17. *213-151. *252-171. *204-34] *253-23. F:.Pef1.):Nr'P=Nr'Q.=n. Nr'Ps=Nr'Qs: P smor Q..= P smor Q Dem. F.*181 33.) F:Nr'P=Nr'Q.n-.Nr'P~i- =Nr'Q-I1 (1) F.-(1).*253-21.) F:. Hp.:):Nr'P=-Nr'Q.=.1i-Nr,'Ps = i-Nr'Qs. [*181L33] —. Nr'Ps = Nr'Qs:.:) F. Prop

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SECTION D] SECTIONAL RELATIONS OF WELL-ORDERED SERIES 37 *253-24. F: Pef.fl PP e f Dem. F.*253-2.*2510.141.41*251-132..-:FHp.P -- ee2,..Nr'PseNO (1) F.*213-32.*251-16.) F: Pe 2,D. Nr'PgE NO (2) F. (1) (2). D: Hp...Nr"Ps e NO.. [*251-122] D. Ps e fl: D F. Prop *253-25. F:.P,QEf i-tA.):Ps D'PssmorQ D'cQs.=. PsmorQ [*253-22. *250-17] *253-3. F: Pefl D. PsEP = P V"P""(1"P = P C CPCCCCP = D"Ps [*213243. *253-13] *253-31. F:. P e I2.: QPsR..Re P PC"P"',P v i P 'PP. QEI R CR"C'R Dem. F. *213245. *253-13.D F:. Hp. D: QJsR. Re CGPs. QeRV'Rc C'R". [*33-24.*2133] R. Re CGPs.! P. Q eR C"R"C'R. [*250315] R. ReP r"PW'CCPvw t'P. ft! P. Qe R ~""R"C'R (1) F. *37-29. *33-24. DF:QeR R "VR1"C'R. f.t!R: (2) [*c13-12] D F: QER C"'R"C"RR. R =P. D -ft! (3) R) P F..(2) Q, R: ReP C"P"C'P. 1P.:) ftP (4) F - (3). (4). R F:RPP~"P"PII C Pvt P.QeR CCR'cCC R..). t!P (5) F. D()(5.F. Prop *253-32. F: P e fL. R C 'Ps.. Ps'R = R C"R0C'R = D'Rs [*213246. *253-13] 253-33. ~F:.Pefl.):Q(Ps D'Ps)R.=2.ReP "PC'CCcP. QeR"CCR'CCCR [*213-247. *253-13] If a is any ordinal number, and P E a, the ordinal numbers of the sectional relations of P are all those ordinals which can be made equal to a by being added to, i.e. all ordinals 3 such that, for a suitable ry, a =13-1-7. (Here y must be an ordinal or 1.) Further, in virtue of *250-67, no member of D'Ps is similar to P; hence, if a is an ordinal, and a=3-1-7, where y +

O, it follows that a + 1. (Observe that a #7 does not follow from 1, Or.a =1.) These and kindred propositions, which are important in the theory of ordinals, are now to be proved. *2534A: P efl - t.=.G'P Q=Qt(aR). P = Qt-R. v..(ax). P = Q f+) x [*21341. *250'13]

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38 SERIES [PART V *253A401. I-: P efl.) P "PC'Pcc v CTP= Q{(aR).P = Q tR.V. (ax). P Q- >x} Dem. F. *253-415.) I-: Hp.!P.:). P "PC'pcl,Pvf=PQ{J(HR). P =Q tR. v.(ax). P Q-F->x1 (1) F.*37-29.)I: P=A.:).P P"C'Pff v t'P t(2) F. *160-14.*33-241. I-:. P = A.:): P = Q t- R. =. Q.R A F. *161-13. *33-241.)D F:. P =A.)::P= Q-f>x.. = Q =A: [*10-24-23]: (ax).P =Q-I+X. =. Q= (4) F.(3).(4).) F:: P =A.:):. (5R). P= Qt-R.v.(ax). P=Q+>x: =. Q=A.F.-(1).(5).) F. Prop *253-402.FPe -'A) D',Ps = Q t(H-R).1 R + A. P = Q - 1?' V. (ax).- P = QfX Dem. F.*253-16 4.:) F::Hp.:):. QeD,'Ps.=E:Q#P:(a[R).P=Qt-R.v.(ax).P=Q-F->x (1) F.*1 61-14. *200-41.) F: Hp. P = Q x.)D. x eG'P. x,-eG'IQ. [*13-14]:) +P(2) F.- *1 60 21.) F: Qt+ P. P = Q t-R.. ft! R (3) F. *160-14. *200-4.) F: Hp.P = Q -R.! R.).! G'Pn C'R.!C'Q~ C']?. F. (3). (4).) F.(1). (2). (5).) F:: Hp.):. Q E D'Pv.=: (RR).R=PA.P= Q4-R v.(ax).P= Q +>x::) F. Prop *253-41. F:.PEf2.QEC'lPs.): (Raa). a e NO. Nr'P =Nr'Q +1 a.. v. Nr,'P = Nr'Q 11 Dem. F. *213-3.) F:. Hp.:): P+~ A: [*20-3-4]): (HR). P=Q tR.V. (ax). P =Q-+x: [*211P283.*200-41]:): (aR). P = Q 4. R. C'Q nC']? = A. v. (ax). P = Q-j>x. x,-. e C'Q: [*180-32.*181-32]): (aR). Nr'P =Nr'fQ -I Nr'fR. v. Nr'fP = Nr'(Q -1 [*251-26])D: (aa). aEcNO. Nr'P = Nr'Q -i-a. v. Nr'P =Nr'Q- i-I:.:) F. Prop *

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SECTION D] SECTIONAL RELATIONS OF WELL-ORDERED SERIES 3 39 *253-42. F: Pel.D). Nr'Pn D'Ps=A [*250-651. *213'141] *253-421. F: Pe1. QED'P,.:). r (QsmorP) [*253-42] *253-43. F:.PE12l.x,yEWIIP.):P~P'lxsmorP~P'y. —.X=y Dem. F.*253-11.) I-: Hp.xPy..(P ~P'fx) Ps(P Pcy). [*213-245]).P P'~x eD'(P ~P'y)s. [*253-421] D. {(P P'x) smor (P ~ P'y)} (1) Similarly I-: Hp. yPx.)D. {(P P'x) smor (P ~ P'y)} (2) [*202-103]).x=y (3) F.(3).-*1 51113.)F. Prop *253-431. I-:Pt-Qe2. ft! Q.).Nr'PtNr'(Pt-Q) Dem. F.*25:3402.)F:Hp.:).PeD'(P4t-Q)s (1) F. (1). *253-421. D F..Prop *253-432. F: P-(+xE!.! P.). Nr'(PtNr'(P-I.~x) [*253-402-421] *253-44. F:a,13e NO - VA.- +Or -D.)a-1-3a Dem. F.*2 5 1. *1 55" 34.) F: Hp.). (a]P, Q). P, Q e 12. a = Nr'P. 138= Nr'Q. ft! Q. [*180-3]:).-(HP, Q).PIQe fl.a =Nor'P.13= Nor'Q.-ftI Q. a +-1= Nr'((P +Q) (1) F. *180-12. *253-431. (*180,01).) F: P, Q e 12. ft! Q.:). Nr'r(P + Q) + Nr'P. [*155-16] D. Nr'(P + Q) + Nor'P (2) F Hp H.). (HP, Q). P, Q e 12. a =N~r'IP.- = Nr'Q. a -j-i +1 N,,r'P. [*13-195]:). a +-13=[a: D)F. Prop *253-45. F:aENO-tfA-t'Or.:).a-1-la [Proof as in *253-44, using *253-432 instead of *253-431] *253-46. F:PefI.Q,RcC'Ps.QsmorR.). Q=R Dem. F. *253>421-16.)DF:Hp. Q= P.D R=Q (1) F. *25316. D F: Hp.- Q + P. R + P. D. Q, R e D'Ps. [*253-13] (a,) X 'PQ=P rX =P c [*253A43.Hp]).Q = R (2) F.(1).(2).) F Prop

40 SERIES [PART V *253-461.: Pe.:. Nr r CGP, eI-+I Dem. F.*253 46.) I: Hp. Q, Re CGPs. Nr'Q = NrR. D. Q = R: D I.Prop *253'462. F:Pe fl.D. Nr I (P C)IP r GI(P e I - I. Nr;P C;P;P C GIP smor~ P C 16P [*253-43] *253-463. F: Pe f.I. Nr; (Ps C D'P,) smor Ps C D'Ps. Nr (Ps C D'P) smor P W (P [*253-462-17-22] *253-47. F: Pe&2-tIA.). Nr'P'PPs = & {(aI(H). a -1-= Nr'P. v * a -a- = Nr'PI [*253,4] *253-471. F: Pefl.). Nre"(D'Ps V t'P) = a' a -1-13= Nr'P. v. a- ia i = Nr'PI [*253-401P13] The following propositions are concerned in proving that Nr'Ps is either Nr'P or Nr'P -1-1. This is proved by using P1 as a correlator. The methods employed anticipate the discussion of finite and infinite series; in fact, when P is finite, Nr'Ps = Nr'P, and when P is infinite, Nr'Ps = Nr'P -- i. But it is important at this stage to know that NrIPS is either equal to or greater than Nr'P, and the propositions are therefore inserted here. *253-5. F: Pefl.). P,;P=P D'P Dem. F. *201-63.c*25-411. F:: Hp.) P = Pi v P2 [*150-11])D:. x (P1 P) w * (5{y, z) xPjy: yPjz. v. yP2z: wP z: [*204-7] (az). xPJw. wPIz. V. (ay, z). xPly. yP2z. wPz: [*250-21-24]:xP w. w e D'P. v. (ay).xPly. y, w e D'P. yPw: [*33'14.*341]:x (P, v PI I P) w. we D'P: [*33-14.*250-242] x, wE D'P. xPw:: FI. Prop *253-501. F: P eL6.PP,;P 'PC Dem. I.*250242.:) F Hp.. P I P = P' P, v P1 PI I P [*71-191.*204-7] =1r aU'P, v (ar'P1)1 P. [*150-1.*50-65]. PI;P = (lPP) 1 P, v (I'P,) 1 P P, [*250-243] =P a P1P,: DF-. Prop

SECTION D] SECTIONAL RELATIONS OF WELL-ORDERED SERIES 41 *253-502. F: PefL). P271'PsmorPCD'P Dem. F. -*253-5. *150-36.): Hp.:). P 2 D'P = P,; (P GP I) (1) F.*151-21.*204-7.)F:Hp.).P,;(P CI'P,)smorP G'PI (2) F.(I). (2).)D F. Prop *253-503. F:PE~. I'P,1=U'P.:). PC GYP smorP D'P [*253-502] This proposition shows that if P is a well-ordered series in which every term except the first has an immediate predecessor, the series obtained by omitting the last term (if any) is similar to that obtained by omitting the first term. The converse also holds, as will be shown later. The hypothesis Pef.GI'IP,=U'P is equivalent to the hypothesis that P is finite or a progression. (Here a progression is not what was defined as " Prog " in *121, but what Cantor calls co; i.e. if 1 E Prog, R,, is a progression in our present sense.) *253-51. F:Pef~.G"P,1=G'P. E!B'P.).Nr'P,=Nr'P Dem. F.*253-2.) F:Hp.P,-,e2r..Nr'Ps = Nr'(P 'P)i[*253-503] = Nr'(P C D'P) - 1 [*204-461-272] = Nr'P (1) F.*21332. F: P e 2r.).Nr'P=Nr'P (2) F. (1).(2). F. Prop *253-511. F:Pef~. UC,'P1=U[,'P..,E!B'P.). Nr'Ps = Nr'P i i.Nr'P C GYP = Nr'P Dem. F. *93-103. *202-552.) F: Hp.). P t D'P = P. [*253503] N.Nr'P G CiP = NrPP. (1) [*253-2]).Nr'Ps = Nr'P - 1 (2) F. (1). (2).) F. Prop *253-52. F: Pe 71. x = minp(IPP - PPI).. OI'P A P'x C I'P,. P1"P'x = P'x. PI1"P'I = P'x - t'B'P Dem. F.*205-14.)F:Hp.). GPAP'1X CPP (1) F. *250'242.:) F Hp.). P'x = P/x '.' P1"P'x [*33-41.Hp] = P1"P'x(. (2) [*72-501.*2047] p. P11P'1X = P'Xr A (PP1 (3) F. (1). D F Hp. Q.'P A P'x = G'P A P'X n 'PI [*1 21 305] = G'P1 A P'x (4) F.(3).(4). DF:Hp.).P1"11P'x- P'Xn P [*33-15. *202'52] = P'x - tfB'P (5) F. (1). (2). (5).) F. Prop

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42 SERIES [PART V *253-521. PE.E P-P)P',P 1 Dem. F. *201P66.)IF: PEI 2. P'xl I. D. xe i'PI (1) -4) F-.(1). Transp.:)F: Hp D.)P'lx 1 (2) F-. (2). (3).:) F-. Prop *253-522. F:Pef~.x=minp'(U'6P-GI'P,).Sz=P1rvrp".) S; (P~ P1P)+= P Dem. FI-.*34,25-26. *50-551.)D F: Hp.:). S;(P ~ PIP) = (P1 r PIx);P ~ lPP w (I ~P*'x);P vi (P1 r P'x) I P I I rP*'"r v I r P1'XI P IP'lx 1P1 -4 4- -4 - [*50-661.*150-36.*305452] = (PI r' PIx);P v P ~ P*'x v' Pi1 ['IP'I P r P*'x vi P*'CxI P r P'a 1IP [*74-141.*253-52. *200-381]= (Pi r P'X);P v P ~ P*'x vi P'rxl PI1 P r P*'x -4 4- -4 4 -[*250-242.Hp] = (pi r' P"X); P v:i P ~ P*'Ix v P'x 1 P rP*'x [*150-36] = (P1;P) ~ P1",P'x w P ~ ', x viPc1 P rP6 [*35-413.*200-381] = P ~ (P'fx v P*'x) [*202-101] = P::) F. Prop *253-53. F:Pefl.x=minp'(GPP-U'PI%).:). Dern. PIEPXwIrP", {P 9-niiOY (P (1P~ F. *20417.*200o381.:)F:Hp.:).PlrP'xvl~rP*'xEl-+l (1) F. *253-52. *50-552.) F: Hp.) CL(Pj rtP'x viI r' P*'x) = (P'x - t'"B'P) vP*x [*202-101] = C'P - BP [*93-103] =CPP [*202-55.*253-521] = C'(P ~ P'P) (2) F.*253-522.:)F:Hp.:).(P~rPIxkwIiP*'x);(P~(PP)=.P (3) F. (1). (2).(3). *15111.)DF. Prop *253-54. F:PeIl2[~!Q['P-([PP.:)-PsmorP~(,PP Dem. F. *250-121. D F:Hp.. E! minp'l(PIP - G'P1) (1) F.-(1).-*2 53-53.) F. Prop

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SECTION D] SECTIONAL RELATIONS OF WELL-ORDERED SERIES 43 *253-55. F: Pefl~j!U'"P-U"P,.:).Nr'Pg=Nr'P-~iDem. F.*253-521. *204-272. DF: Hp. D. Pe%.,e 2r() -.(1). *253-54-2. DI F.Prop *253-56. F:.Pefl.):U['P1=U'"P.E!B'P.).Nr,'P,=Nr'P: [*2-3-51 511 55] *253-57. I-:PEIEQ('fP1=PlP.E!B'P.:). j-Nr'lP=Nr'Pi- iiNr'PtNr'P Dem. F.-*253-51. FI: Hp.:). Nr'PA= Nrl'P. [*253-21])1 —Nr'P = Nr'P+ —i 1 [*253-45] D. 1 —Nr'P + Nr'P (2) F. (1).(2).) F.Prop *253-571. F:Pefi.-.. (c1'P1=PIP.E!B'P).).i-jNr'P=Nr'P Dern. F.-*253.56.) F:Hp.D. Nrl'Ps= Nr'lP-i [*253-21].jiNr'P i-= Nr'fP-i [*181P33]).1iNr'P Nr'P:DIF. Prop *253-572. F: Pe&2 -t'A., (PIP1=U'P.E!B'P).:).1i-~Nr'P#Nr'P-~i[*253-571P45] *253-573. F:.Pef~.:):([P, =PIP.E!B'P.E=.i4.Nr'PtNr'P [*253-57-571] *253-574. F:. P e f- t(A. ~:(I'P, = UP. E! Bcp'P.. Nr'lP-=Nr,'P-j [*253-57-572]

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*254. GREATER AND LESS AMONG WELL-ORDERED SERIES. Summary of *254. In the present number we have to prove that of any two well-ordered series one must be similar to a sectional relation of the other. From this it will follow that of any two unequal ordinals one must be the greater. The propositions of the present number are due to Cantor*. Our procedure is as follows. We define a relation "RPmQ," meaning "R is a proper section of P, and is similar to Q," i.e. RPsm Q. -. R e D'Ps. R smor Q. In virtue of *253*46, if P, Q e f2, Psm e 1 - Cis (*254-22) and Pm rD'Qs 1-,1 (*254-222). Thus if S is any proper section of Q which is similar to some proper section of P, the proper section of P to which it is similar is Pm'S. It is easy to prove that Pm; Qs C D'Qs is a section of P; and if D'P, C G('Qsm, i.e. if every proper section of P is similar to some proper section of Q, we shall have (*254'261) PsD'P= Psm; QstDQs. Hence it follows (*254'27) that if, further, D'Qs C I'PSm, we shall have Ps DPs smor Qs DQ, i.e. by *253-25, P smor Q (*254-31). Thus (A) if every proper section of P is similar to some proper section of Q, and vice versa, then P is similar to Q. Consider next the case in which every proper section of P is similar to a proper section of Q (i.e. D'PS C ('Q) m), but not vice versa, so that [! D'Qs - (CPBm. It is easy to prove that, under this hypothesis, if Se D'Qs - (PPsm, then D'Ps C ('sm (*254032). But if S is the minimum (in the order Qs) of the class D'Qs - ('Pm, then D'Ss C ('Psm. Hence, by (A), SsmorP (*254-321). Thus (B) if every proper section of P is similar to a proper section of Q, but not vice versa, then P is similar to a proper section of Q (*254'33). * Math. Annalen, Vol. 49.

 From (B), by transposition, we find that if every proper section of P is similar to a proper section of Q, but P itself is not similar to any proper section of Q, then every proper section of Q is similar to a proper section of P, whence, by (A), P is similar to Q (*254'34). Hence, if there are proper sections of P which are not similar to any proper section of Q, the smallest of such sections (say P') must be similar to Q, since it is not itself similar to any proper section of Q, but all its proper sections are similar to proper sections of Q. Hence (C) if there are proper sections of P which are not similar to any proper section of Q, then there is a proper section of P which is similar to Q, i.e. F: P, Q e f. a! D'Ps - (I'Qm.. Q e I'Psm (*254-35). Thus either (1) a! D'P,- 'Qm, in which case Q e ('Psm, or (2) g!D'Qs-('Psm, in which case Pe(I'Qm, or (3) DPs C (IQm and D'Qs C ('Pm, in which case, by (A), Psmor Q. Thus (D) if P and Q are any two well-ordered series, either they are similar or one is similar to a proper section of the other (*254'37). We now proceed to define one well-ordered series P as less than another well-ordered series Q if P is similar to a part of Q, but not to Q, i.e. we put less = PQ {P, Q e f. a! R'Q n Nr'P. - (P smor Q)} Df. (Observe that we have R1'Q in this definition, not D'Qs.) It follows from (D) that, P and Q being well-ordered series, if P and Q are not similar, one must be less than the other (*254'4). It follows also from *250 65 that if P is similar to a proper section of Q, Q cannot be less than P (*254'181). Hence P is less than Q when, and only when, P is similar to a proper section of Q, i.e. Pless Q.. P, Q e. P e Qsm (*254'41). Hence if each of two well-ordered series is similar to a part of the other, the two series are similar (*254'45); and in any other case, one of them is similar to a proper section of the other. From the above results we easily obtain the following propositions, which are useful in the ordinal theory of finite and infinite. *254'51. F: P less Q.. P, Q e f. RI'P n Nr'Q = A I.e. one well-ordered series is less than another when, and only when, no part of it is similar to the other. *254'52. F: P e 1. a C C'P. a! C'P p'P"a.. P a less P I.e. any part of a well-ordered series which stops short of the end is less than the whole series.

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46 SERIES [PA- *254-55. F:.QlessP.=2:P,Qe6fZ:(aR). RsmorQ.RC-P. a! C'Pnp'P"6C'R I.e. one well-ordered series is less than another when, and only when, it is similar to a part of the other which stops short of the end. *254-01. less =PQ tP, Q e! Rl'Q nNr'P. -.(Pswnor Q)} Df *254-02. PSM = (D'Ps) 1smor Df *254-1. F:P less Q.= P, Q E1. a! Rl'QriNr'P. (Psmor Q) [(*254-01)] *254-101. F:P, Qef2.PC-Q.,,(PsmorQ).:).PlessQ [*254'1] *254-11. F: RP,mQ.=.RBEDI. BRsmor Q [(*254-02)] *254-111. F. PSM'Q =D'PsnNr'Q [*254-11] *254-12. F- Q EU('Psm,,.. a! D'Ps n Nr'IQ [*254-1 11] *254-121. F D 'Ps C G"Ps~m [*254-12. *152-3] *254-13. F:. P smor P'. Q sm or Q'.)P less Q. P' less Q' [*151-15. *152-321. *254-1] *254-14. F:S eD'Q. Te P simio-Q)T;S eD'Ps nNr'S Dem. F.*l5O-37.)F:Hp.S=Q.)8-. T;S=(T;Q)~T"/3 [*151-11] = P ~TIfl, (2) F. *212-7.)F:Hp.$8Esect' Q.D. T"I8,6sect'P (3) F. *37 43. D F:Hp.I8e sect'Q -t'A D.)a! T"/3 (4) F.-*150-22. D F: Hp. T",89=COP. D. T"W3= T"G'(Q: [*72-481])F:H p.T"/3 = C'P.,&Esect'Q.).I3= CQ [Transp] F F: Hp fle sect' Q - t'GIQ. D. T "O 3 t0P (5) F. (3). (4). (5).) F: Hp.fE sect'Q - fA - tfG'Q. D. T"/36e sect'P - t'A - 'GP (6) F. (1). (2). (6) F F: Hp. D. (Ha). ae Esect,'P - tA- tIC'P. T;S= Pi a. [*213-141]:). T;SeD'Ps (7) F. *151-21.)F:Hp.). (TS) srnor S (8) F.(7). (8).) F.Prop *254-141. F:P smor Q.).D'Qs C([",mD,'Ps C IQsm Demn. F.*254-12-14.:)F:.Hp.:):SeD'Qs.:).SeUI'Psm (1) F. (1). *a1514.) F. Prop *254'142. F:BRe C'Pis)RSM C POM Dem. F. *213-241.:)F: Hp.:). D'B C D'Ps (1)

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SECTION 1)] GREATER AND LESS AMONG WELL-ORDERED SERIES 4 47 *254-143. F: Q E U'Psm.:). C'Q, C G'Pgm Dern. F. *254-12. D F::Hp. D. (2jR).Re D'Ps.. Rsmor Q [*25-4-141]).(H1R)..1 ED'Ps. D'Qs C U'1?s. [*2-4-142]1 D'fQs C U'Psm. [*213-16.Hp]).Q ~1"(sect'Q - t'A) C (I'PSm [*213-1])*C'Qs C [PPsm: D F. Prop *254-144. F:P=~A.:).Psm=A [*213-3.*2.54-11] *254-15. F:.Qp0C-J.H!B'P. Pp CJ.): QEU'lPsm.-=.C'IQSCU~Psm Dem. F. *2 54 143.)F: QECap~sm.. of'Qs C ['CPSM (1) F. *213-142. *211'26. D F:.Hp. ft! Q. D): QE E6s [*22-441] D: C'rQsCU'Psm.). QeCVPsm (2) [*200'35] D. A e P ~ "(sect'IP - t A). [*213-16].AE D'Ps. [*254-121])AeG'l Psm (3) F. (1). (4).)D F. Prop *254-16. F:. Qsmor Q'. D: P mQ=P3m'Q': Qe'PTSM. r. Q'ePGs Demn. F. *254-111. *1 52-321.)F: HpP.): mQ = Psm'Q':(1 [*13-12]):jSM'Q..-!Psm'Q': [*33-41] D: Q eU' PSM. Q'EU'Pm (2) F. (1). (2). DF. Prop *254-161. F: P smor P'.)D. G'PSM = GTS Dem. F.*254,14. D F: T eP srnor P'. Se D'P's nNr'Q. D. T;SE D'P n Nr'Q: [*254-12]:)F:T eP s-i'or bP'.Qe ('P'm.:). QE 'Psm. [*1 51-12] D F: P smor P'. D. C'P'SM C G1I'PSM (1) F.(1). *151-14.:)F:P srnior P'. G 'P,, C G 'P' (2) F.(I). (2). D F. Prop *254-162. F:. PsmorP'.QsmorQ'.:):QEU'"Psm.=.Q'eU'"P'sm [*254'16-161] *254-163. F:RE U'Qsm.:). U'RsmCC,' Qsm Dem. F.*254-12.:) F: Hp.:).(HjS).RsmorS.SeD'lQs. [*254-161-142]).(2jS). U'R~m = Q'ssm a G'SSM C G'QSM. [*13-195] C).URSM C U'IQsm: D F. Prop

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48 SERIES [PART V *254-164. F: D'Ps CP(Qsm.). D'Ps= Psm"I(D'fs Q (IU"m))= "D ''DQs Dem. F. *254-11.)F: Hp. RE ED'P.)(HS). SE D'Qq. RBsmor S. [*254-11])(aS). SeD' Qs.BPsmS. [*37-1] RBE PSM"cD'Qs (1) F. *254-11.)F. PSm"1D'1Qs C D"Ps (2) [*37'26] -P~m "(D 'Qs~ rn GIPs.):) F. Pr op *254-17. F: P6fl.QED'Ps.RC-Q.).r-%}BsrnorP) Dem. F. *204-21.)D F: P E. R CE P. B smor P. D. R e Ser. [*204-41] R.B= P~ C' (1) F. *250-65. Transp. [*211-133-44]).(HQ). Q E P~ " 1(sect'P - tICG'P). R c. Q. F. (3). Transp.:) F. Prop *254-18. F: QcD'Ps.).-r (P less Q) [*25-4'17-1] *254-181. F:QEGf'PSMO.).r-(PlessQ) Dem. F. *254-18-12.) F:Hp.).(HR).RBsmor Q. -(P lessRB). [*254-13] D.. (P less 9)::) F. Prop *254-182. F:P f2. Qe6D1Ps.). QlessP [*254-101.*253A42118] *254-2. F:P cfL.QEGI"Psm.Q less P Dem. F. *254-11.:) F: Hp.).(R).BRe6DIP,. RBsmor Q. [*254-182]).(HB). B less P. B smor 9. [*254-13] D. Q less P: D F. Prop *254-21. F:Pe2. QE6[Psm.BC-Q.B,6fl.).BlessP Dem. F.*254-12.:)F: Hp.).(:qS, T).S6D'P. T~eS:~hi6rQ. [*151P21.*150-31]:). (21S, T).Se,6D'"P,. TE Ssgm-6rQ. T;Rsm orR. TBRCS. [*254-17] D..(T). TBR sm or B. T;B C P. (T;BR sm or P). [*151.17]:). (a T). T;B smorBR. TBR C P. '-(B smor P). [*254-1]:) BRless P:) F. Prop

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SECTION D] GREATER AND LESS AMONG WELL-ORDERED SERIES 49 *254-22. F: Pef2.:).PsmE61 -+ CIS Dem. F.*2 54'11.. F:. RPsmQ.-SPmQ.):?, S eD 'Ps. RsmorS8: [*2-3-46])Pe i2.:).R?= S (1) F. (1). Comm.:) F-. Prop *254-221. F: Pefl.).EL'P81MCf Dem. F. *254-12. *253-13. I-:Hp. QEG'IPsm.).(gR,a).R=Ptca. RsrnorQ. [*250-141.*251-111] D. Qe E2 D F. Prop *254-222. F:P, Q 12.). P,,m [D'Qsc I —+1 Demn. F. *254-11)F:. R(Pm rD'Qs) S. R (Pm rD'Qs) S'.) 84, S',e D'Qs. R smor S. R smor 5': [*2-3-46]:Q6fl. D. S= S'() F. (1). Comm.:)F: Hp.:). Psm rDQs eCls ->1 (2) F. (2). *254-22.:)F. Prop *254-223. F CIIv'(Psm r D'fQs) = Qsmrn D'Ps Dem. F.*254-11..) F:R(Pmrn D'Q,)S..RcD'J, Se D'Qs. Rsmor S. [*151-14] S c SDQ0. R eD'Ps. S sror R. [*254'11] S8 (Qsm r DfPs) 1?::) F. Prop *254'224. F:Q fa E!PSM'S.Se D'Qs)S = Qsm'Pm 'S Demn. F.*204-223.)F:. Hp.:): SQm (Psin'S).= (P,,m'S) PsmS (1) F.(1).*30-32.*254-22.) F. Prop *254-23. F:P E &.Q9E IPsm.* P,,mjQ =t'(D'P n Nr'Q) [*254-22-111] *254-24. F:P,QE&E. ReD'AAG'fQsin.SeRI'RAD'Ps.).SeU'"Qsm Dem. F. *213-24.:) F:Hp.). S e D'1Rs. [*'254-143.Hp] S c U'Qm111 D F. Prop R. &W. Ill. 4

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50 SERIES [PART V *254-241. F:. P cf~. Q, ReG'lPs) ReG(1'Qsm. R E.RD'Qs Dem. F. *254-121.:) F::REcD'Qs.).1?eU'Qsm (1) F. *254-142.)3 F: lip. QeG'JIXs.) D. Qsm.CJRSM (2) F.*253-42. DF:R,6f2..Rr~ceUR~ (3) F. (4). Tran sp. (3).)D F: Hp. 1? E U'Qsm.).D Q ORG'11. Q [R* [*213-245]).. (QPS-R). Q ~ R. [*213-153.

Hp]) RPsQ. F. (1). (5).) F. Prop *254-242. F: Qe f2. T cP giii-or Q.Se D'Q.)D. T; S8= P,/S Dem. F. *254-14.:) F: Hp.:)..T;Se6D'P nNr'S. [*25-411] (T. (S) PS S [*254'22.*259"1*1l])*T;S =PSiMS:) F. Prop *254-243. F: QEf2.SED'Qs. TEPSiYiforS.S' QsS.).T;S'=Psm'S' Dern. F.*213-245.*253'18.) F:Hp.).SeIf2. S'eD'S,. [*254-242] D. T;S'==Ps 'S':)DF. Prop *254,244. F:P,Qeli1.SeD'Qs A (I'Psm. TE(Psm'S)-m-orS.S'Q~S.:). T;S = PSM"SS T;S'=- Psm'S'. (T;S') Ps (T;S) Dem. F.*2054'243.) F:Hp.-R = P,,m'S.T). = g (1) F.*254-11. DF:Hp (1).:).1ReD'Ps. (2) [*25-4142]:). RS PsMn (3) F. *151-11.)F:Hp (1).).R = T8. (5) [(2)])T;SeD'"Ps (6) F. (1). (5). *254-11. D F: Hp (1).) T;S' c D'(T;S) (7) F. (6).(7). *213-244.:) F: Hp (1).).(T;S') Ps (T;S) (8) F.(9). (4). (8).) F. Prop *254-245. F: P, Q C 2 Se D'Qs A Q'Psm, S'QsS.:). (P,,'S') PS (Psm'S) Dem. F. *254-22'11.:) F: Hp. D. (Psm'S) smor S (1) F. (1). *254-244. DF..Prop

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SECTION D] GREATER AND LESS AMONG WELL-ORDERED SERIES 51 *254-25. -:. P, S'e D'Q, ('Psm. ~: S'QS. -.(Psm'S') P (PPsmS) Dem. F. *254-245.:. Hp. D: S'QS.. (Psm'S') Ps (PsmS) (1) P.S, P PsmS', P, Q.(1) S, S, Q,P F:. Hp.: (Ps,'S') Ps (Psm'S). D. (Qs'Psm'S') Qs (Qsm'Psm'S). [*254-224] D. S'QsS (2). (1). (2). 2) F. Prop *254-26.: P, Q e f. D. Qs: (D'Qs n I'Psm) = Qsm;(Ps D'Ps) Dem. F. *254-25.:: Hp.:. S' {Q, (D'Qs n 'Pm)} S.-:, S' e D'Qs n a'Ps. (Psm'S') Ps (Psm'S): [*254-22]: S, S' e D'Qs: (JR, R'). RPmS. R'PsmS'. R'PR: [*254-223]: (3[R, R'). SQsmR. S'QsmR'. R, R' e D'P. R'PsR: [*150-11] -: S' {Qs;(Ps C D'Ps)} S:: F. Prop *254-261. F: P, Q e fQ. D'Qs C (aP,,11.. Qs C D'Qs = Qsm;(Ps C D'PS) [*254-26] *254-27. F: P, Q e f. D'Ps C G'Qsm. D'Qs C (Psm.). Qsm r C'(Ps I D'Ps) e (Qs: D'Qs) smor (Ps C D'P,) Dem. F. *254-222.) F: Hp.. Qm [C'(P, D'P) e 1 - 1 (1) F.*37-41.) I: Hp. 3. C'(P D'P,) C (I'QS (2) F. (1). (2). *254'261. *151-22.). Prop In virtue of the above proposition, we have, when its hypothesis is realized, (Qs C D'Q,) smor (Ps C D'PS), whence, by *253'25, Q smor P. This proposition is the converse of *254'141. In the above proposition we take Qsm C'(Ps C D'Ps) as the correlator, rather than Qsm r D'Ps, so as not to have to make an exception for the case when P e 2r. For if P e 2r, D'Ps e 1, but Ps D'Ps = A. Thus Qsm D'Ps is not a correlator in this case. The following propositions, down to the end of the present number, are important, and give the foundations of the theory of inequality between wellordered series and between ordinals. 4-2

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52 SERIES [PART V *254-31. F:P, Q. D,'P, C U'Qm..D,'Q, C PPsrn.. P smor Q Dem. F. *254-27.)FHp.): (P, ~ D'Ps) smor (Q, ~ t~) [*253-25]): f! P.t! QD.).P smor Q (1) F. *254-144.)D F: Hp. P = A.:). D'Qs A. [*213'302])Q=A. [*153-101] P. smor Q (2) Similarly F: Hp. Q=A.). PsmorQ (3) F.-(1). (2).-(3).) F.Prop *254-311. F:. P, QEf.)D: D'P C (1'Qm. D'Qs C (IP~,.. P stuor Q *254-32. F:P, Q e2.D'Ps C U'Qsm. S eD'Qs - U'Psm.:).D'Ps C GI'51n Dem. F.*254-24.)F:Hp. R, S'e DQs. S'C-R. ReG'1Psi.). S'e (P,,m (1) F.(1). T ransp. D F: H p. RE D'1Q, (1'Psm.) (S C.

R). [*254-22-11.*213-245]:).(P,,m/R) smorR?. RE~D'Ss. [*254 12]:). (Psm'R)c E1s. (2) F. (2). *3761. D F: Hp.:). PSM6"(D'Qs r ClPPm) C (ISsl [*254-164] D. D'"Ps C PISm: D F.- Prop *254-321. F:P,QefL.D'Ps CU'Q8I,. S=min(Qs)'(D'Q, -IPsm).).SsmnorP Dem. F. *2035 14. F: Hp.. Q'sc C G'PSm. [*213-246] D. D'(Ss CGCPsm (1). *254-32. DF:Hp. D.D'Ps C P18sm (2) F. (1).(2). *25 4-31.:) F. Prop *254-33. F:P,Qefl.D'PsCPIQsm.a!D'Qs-U'"Psm.:).PeI'fQsm Dem. F. *253'24. D F: Hp.).E! min (Q,)'(D'Qs -U'Psm). [*2'4-321])(aS). S eD'IQ, 8Ssmor P [*254-11] D. Pe E IQsm:)D F. Prop *254-34. F:P,Qefn.Pr-,eU'Qsm.D'PsCU'"Qsm.).PsmorQ Dem. F.*25'4-33. Transp.:)F:Hp.).D'Qs C U'Pm. D'Ps C U'Qsm. [*254-31]:). P smor Q: D F. Prop

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 *254-35. F:P, Q e!D'(Q - G'Psm.)PEWIQsJm Dem. F. *253-24.:) F: Hp.-). E! min (Qs)'(D'1Qs - IP1) [*205-14]).(3S). S ED'Q, - (1'PSM. QS'S C G'Psm. [*213-246])(2[S).S eD'cQ, - GL'P~,,,. D'"Ss C (T' PSM [*254-34].([S). Se I)'IQ,. S smor P. [*254411])P E (OiQ,,,::) F. Prop *254-36. F: P, Qef~. a! D'Q~ - U'Psi).O'rP~ C P (Qs [*254-35-143] *254-37. F:.P, QEf~.:):PstnorQ.v.PEEI'Qsm.v.QECU'Psm, Dem. F.*254':3l.:F: Hp.D'IsCPIQsm.D'QsCllp~sm.).PsmorQ (1) F.-*254-35. DF: Hp.j! D'Q, -U'lP,,m.n)PeG'FQSr (2) F. *254-3-5. F:Hp. g! DYP - U'Qqi. m. DQ eU'Psm (3) F.-(1).(2). (3).)F.Prop This proposition is the most important on the relations of two wellordered series to each other's segments. It shows that of every two well-ordered series which are not similar, one must be similar to a segment of the other. *254A4. F:. P,Qcfl.): Pless Q.v. PsmnorQ. v.Qless P Dern. F.*20542.:)F: Hp.Pe'Qsm.).PlessQ (1) F.*254'2.:)F:Hp. Q eU'1Psm.).Q less P (2) F.*254-37.)DF: Hp.P' eU('Qsm.. Qr -'eU'Psm..). P smor Q (3) F. (1). (2). (3).:)F. Prop *254-401. F:.P,QefL.:):ess'P-less'Q.=.PsmorQ Dern. [*254-4])P smor Q(1 F. *254-13. F: Hp. Psmor Q.).less'P= less'Q (2) F.(I). (2).)F. Prop *254-41. F: PlessQ.=-.P,Qe92.PEPIQsm,.=-.Qd2.PECPQsm Dem. F. *254'2. F F:Qe &2. PEEUIQsm D.)P less Q(1 F. *254-181.)F:QeU'IPsm, D.. (P less Q) (2) F. *253-421)F:Q e f. R eDQ. P smor R. D.r(P smor Q) [* 2-5 4 -11]) F Q e f.PeU' P,,m,. D. (P smor Q) (3) F.(2). (3).*254-4.) F:Q e. PeU'IPsm.. DP less Q (4) F.(). (4). D F:Pless Q..Q6 2.P eP1Qg. [*254-1].P, Q e I. P e PQ.am,: DF.Prop

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 *254'42. F. less C J. less2 C less Dem. F. *254-1. D F: P less Q. D., (P smor Q). [*151.13] P+Q (1) F. *254-163. -: R e ('Qsm. S e'Rs. *.S (I'Qs: [*254-41] D F:RlessQ. SlessR..SlessQ (2) F. (1). (2). D F. Prop The relation "less" fails to generate a series, because it is not connected, two similar well-ordered series being neither greater nor less than each other. On the other hand, the relation Nr;less is serial, since two similar wellordered series both contribute the same term to the field of Nr;less, and therefore connection does not fail. The relation Nr;less will be dealt with in the next number. *254'43.: Q e fL- 'A. D. A less Q [*254'1.2504. *152l11] *254-431. F. ('less = fl - tA. C'less C 12 Dem. F.25443. 3 D: Q e f - l'A. D. A less Q (1) F. *2541. *25'13. F: Q =A. D. Q e Cless (2) F. *2541. D. C'less C (3) {-. (3). (2). Transp. D F. (less C fI - I'A (4) F. (1). (4). D. (Cless = f - L'A (5) F. (3). (5). D F. Prop In order to obtain C'less = Q, we need, as appears from (1) in the above proof, [! 2 - t'A. In virtue of *251'7, this requires a! 2. By *101'42'43, this holds if "less" has its field defined as belonging to a class-type or a relation-type. If, however, "less" has its field defined as composed of individuals, the primitive propositions assumed in the present work do not enable us to prove at! 2, nor therefore to prove a! less. It should be observed that "less," like "sm" and "smor," is significant when it is not homogeneous; but "C'less " is only significant for homogeneous typical determinations of "less," because only homogeneous relations have fields. *254-432.: 2,. =. g! less too'a tooa. =.! - 'A n to' Dem. F. 251-7.): g! 2a. =.g! t - iA n tooa. (1) [*254j43] -. ([Q). Q e - t'A n to' a. A less Q. [*55-37] D. ({Q). A less Q. A, Q C t'ca T ta. [*553] D. g! less A too'a ' too'a (2)

SECTION D] GREATER AND LESS AMONG WELL-ORDERED SERIES 55C F. *35103.) I-!: less Ai t0'a Tt t..). (HP, Q). P less Q. P, Q t,)c" [*254-431]! f2 - t"A n Wa [(1)] a2 (3) F.-(1). (2). (3).)l F- Prop *254-433. F. ft! less A t,,'Cls T t,,'Cls. ~! less A to'Rel T to'Rel [*254-432. *1 0142-43] *254-434. F:' less.C'less = f2.. B'less= Dern. F.*250-4.*33 24.:FC'less=f2.).) f!less (1) F.*93102. *33-24. D F: B'less = iA.). t! less (2). F.*254-43. D F: Q efn - L'A. D.)Aless Q (3) 1 (3). D F: q! f2 - tIA. Ac i D'Iess. [*i254~4431] A = B'less (4)) F. (4).*254-431.)F: a!12 - 'IA. D.. 'less = 2 () F. (1) - (2). (4) - (5). D)F. Prop -* 4~~*254-44. F P P C'less.). C'less - less'Pu Nr'P v less'P Dern. F.*254K13. D)F: Hp..NrP C Cless (1) -~~ 4~F. (1). *33 152.2- DF: Hp. D. less'P v Nr'P v less'PCC'less (2) F. *2554-1.)F. Oless C fl. -+ 4~~[*254-4]:) F:. P e C'less.): Q E C'Iess. D. Q e less'P v Nr'P v less'P (3) F. (2).(3). DF.Prop *254-45. F:P,Qef1. 1!Rl'PANr'Q.g!Rl'Q R Nr'P.).PsmorQ Dem. F. *254-42. D)F: P less Q. 'h(Q less P) (1) F.*254-1.)F:P,Qe 7l.3!RL'QANr'P. N (PsmorQ).).PlessQ. RIA D ~. (Q& less P). [*254I1.Transp]). r! RI'P n Nr'Q (2) F. (2). Transp. F1. Prop This proposition is the analogue, for ordinals, of the Schr~ider-Bernstein theorem.

56 SERIES [PART V *254-46. F: Pless Q P, QE!R'A rP ~ RI'Pnr Nr'Q Dern. F.*1 5211. *61 34.) -:P, Qe ~.{ a! RI'IQ Nr'P.,r~ a! Rl'P Nr'Q.:). P, Qc f~. aj! RI'Q Nr'P.,. ~(P smor Q) [*25441])P less Q (1) F. *254-145. Transp.)D F: Pless Q. D. P, Qe&Lf.!R1Q nNr,'P.,-r:ff! R1'P ^Nr6Q (2) F.-(1). (2). IF. Prop *254-47. F:P e fZ.). Ps=less ~C'1P, Dem. F.*213-245.F:Hp.) RPsQ.~.ReD'Qs. QeO"Ps. [*254-121]). e Ui'QSm [*254-41] R less Q(1 F.- *254-181. Transp.) F: Hp. Q, B e G'P,. Bless Q.:). Q,., e1U'Bsm. [*254-121]:).QJED'Bs (2) F.(2).*213-25.*254-42.)F: Hp. Q,BEC'Ps.RlessQ.).BED'Qs. [*213-245] D. RPsQ (3) F.- (1). (3). D F. Prop *254-

5. F:. P, Qa62.)D: IR1'P A Nr'Q = A.E. ~!Rl'Q n Nr'P.-, (P smor Q). =. P less Q Dem. F *254-46.)F:Hp. Rl1P nNr'Q =A.).(Q less P) (1) F *61-34. *152-11.)F: Psmor Q.)D. Pe Rl1PnNr'Q (2) F.(2). Transp. D F:Rl'P fNr'Q =A..(Psmor Q) (3) F. (1). (3). *2-4-4. DF: Hp. Rl'1P Nr'Q = A. D. Pless Q (4) F.*254-46. D F: Pless Q.). Rl'1P fNr'Q =A (5) [*254-1] =-. a! RII"Q n% Nr'P.(P smor Q):. D F. Prop *254-51. F: Pless Q.=-.P, Qefl. R1'PeNr'Q= A [*254-51] *254-52. F:Pef2.caCCGP.a! C'Pnp'"Plla.:).P~alessP Dem. F.*250O141.DF:Hp.D.P~aefi1 (1) F.*250O653. DF: Hp..(P a smor P) (2) F.-(1). (2). *254 101.)D F. Prop

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 *254-53. H: P,QefL Qcp.a!G'Pnp'P"C'Q.).QlessP Dem. F.*250O652. IF: Hp)'(Q smor P)(1 FI-(1). *254'101.:) F-.Prop *254-54. F:P,Qef2.RsmorQ.BC-P.Hj!C'Pnp4P"U'R. 2. QlessP [*254-53-13] *254-55. I-:.QlessP.=:P,Qef2:(aR).Rsmor Q.1?CP.! C'Pnp'P"G'IR Dem. F. *254-41.:)F:. Qless P.):P, Q E 2(a]).]?smor Q.RL 1? e [*213-18]):P, Qef~:(a[R).RsmorQ.]?C-P.a!CG'Pnp'P"'C'R (1) F. (1). *254-54.:) F. Prop

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*255. GREATER AND LESS AMONG ORDINAL NUMBERS. Summary of *255. If P and Q are well-ordered series, we say that Nr'P is less than Nr'Q if P is less than Q. Thus if u and s are ordinal numbers, we say that p is less than v if there are well-ordered series P, Q, such that z = Nr'P and v = Nr'Q and P is less than Q. In order to exclude the case where, in the type concerned, we have Nr'P=A or Nr'Q = A, we assume p = Nr'P and = Nor'Q. Thus we put < v. -. (aP, Q)., = Nor'P. v = Nr'Q. P less Q, i.e. we put < = Nr;less Df. In order to be able to speak of Nr'P (where the type of "Nr" is left ambiguous) as greater or less than Nr'Q, we put <Nr'P.=., < N Dr'P Df, Nr'P < p. =. Nor'P < p Df. The treatment of types proceeds, mutatis mutandis, as in *117, to which, together with the prefatory statement in Vol. II, the reader is referred for explanations. In virtue of *254'46 and *117'1, there is a close analogy between cardinal and ordinal inequality. That is to say, most of the properties of cardinal inequality have exact analogues for ordinal inequality, and these analogues have analogous proofs. (In the present number, when a proposition is analogous to the proposition with the same decimal part in *117, and has an analogous proof, we shall omit the proof.) But ordinal inequality has a good many properties which have no analogues for cardinal inequality. The chief of these, upon which most of the rest depend, is *255'112. F:. /., v e NoO. ~::p < v. v. p = smor". v. v <E p where "NoO " stands for "homogeneous ordinals," i.e. NO n NoR. We have also, what is often important, *255-17. F: Nr'P>Nr'Q. -. Q less P. -. P, Qe f. Qe ('Psm.. P, Q e. a! D'Ps n Nr'Q

SECTION D] GREATER AND LESS AMONG ORDINAL NUMBERS 59 so that *255'171. F:. P e l.): /, < Nr'P. =.. e Nr'D'Ps - t'A and more generally, *255'172.:. Pe l.D: At < Nr'P.. (aa). a C'P. t! C'P n 'P"la. N = Nr'P P a. a!, As in cardinals, /. is greater than v if (and only if) u/ is the sum of v and an ordinal other than zero, including i except when v = 0r (*255-33). But it is necessary to the truth of this proposition that the addendum should come after v, not before it; i.e. v i+ O > v unless w = 0r (*255-32'321), but 4- v is often equal to v. If a, /3, 7 are ordinals, and a >,/, we shall have y7 - a >y + (*255'561), a /3>38 if a+Or,./3Or (*255-571), a X 7 >/,3Y if 7 Or (*255-58), 7ry X'> y if y is of the form + i (*255-573), 7yXa> yx,8 if 7 is of the form &i- (*255-582). From the above propositions it follows that if a, i/, y are ordinals, y- a = 7 4-,/3. D.a =3 (*2.55'565, where /3 may be substituted for smor"fL whenever significance permits; cf. note to *120'413), which gives the uniqueness of subtraction from the end (subtraction from the beginning is not unique); aX y = iry. D. a = 3 unless y = 0r (*255'59), which gives the uniqueness of division by an end-factor; y7 a= r. D. a=,8 if 7e=4-1i (*255-591), which gives the uniqueness of division by a beginning-factor of the form 8ii. We do not have generally a, /3, y e NTO. a <..). a expr 7 < / expr 7, because a expr y and 3 expry are in general not ordinal numbers, since series having these numbers are in general not well-ordered. Thus the theory of ordinal inequality has only a restricted application to exponentiation. This subject cannot be adequately dealt with until we have considered finite and infinite series. If a is an ordinal, C"a is the corresponding cardinal, i.e. the cardinal number of terms in a series whose ordinal number is a. Thus the cardinal numbers of classes which can be well-ordered are C"'NO, i.e. *255-7. F. Nc"C"12 = C"'NO

60 SERIES [PART V It is evident that *255-71.: P less Q.. Nc'C'P < Nc'C'Q whence, by *254'4, *255-73. F:.P,Qe2.:): Nc'C'P < Nc'C'Q. v. Nc'C'P = NcC'Q. v. Nc'C'P > Nc'C'Q whence also *255'74.:. a,/,8 C"'NO - t'A.: a 3. v. a> Thus if two classes can both be well-ordered, they either have the same cardinal, or the cardinal of one is less than that of the other. We have *255'75. I: P, Q e f. NcC'P < Nc'CQ.. P less Q or, what comes to the same thing, *255'76. F: a,/, eNO. C"a < C"R. D.. a < The converse of this proposition only holds for finite ordinals. If a is an infinite ordinal, a - i always exists and is greater than a, but C"a = C"(a - i). (The existence of a + i is deduced from that of a by taking a member of a, and removing its first term to the end. The result is a series whose number is a + i, in virtue of *253'503'54.) *255'01. < = Nor;less Df *255'02. > = Cnv'< Df *255'03. NO = NO n NoR Df Thus "NoO" means " homogeneous ordinals." In virtue of *155'34'22, this is the same as "ordinals other than A." It is not, however, strictly correct to put N00 = NO - t'A, because if the " NO " on the right is derived from an ascending Nr, it will not contain all the ordinals in the type to which it takes us, but only those which are not too big to be derived from the lower type from which "Nr" starts. Thus in this case NoO will be a larger class than NO- t'A. If,

however, the "Nr" from which the "NO" on the right is derived is homogeneous or descending, we shall have NoO = NO- t'A. *255-04. < = < w smore NoO Df This definition leads to the usual meaning of " less than or equal to." We want the relation "less than or equal to " to hold only between numbers of the sort in question (cardinal or ordinal), and we want "equal to" to hold between two numbers which are merely different typical determinations of a given number, provided neither of these typical determinations is A. That is, if A is an ordinal which is not A, smor",/ is to be reckoned equal to / in every type in which it is not A. Thus if v = smor"/, i.e. if v = smore'/, we

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SECTION D] GREATER AND LESS AMONG ORDINAL NUMBERS 61 shall reckon v equal to p if both are ordinals and neither is A, i.e. in virtue of *155*34*22, if,, v e N0O. This leads to the above definition. *255-05. > = Cnv'< Df *255 06., < Nr'P. =.p < Nor'P Df On this definition, compare the remarks on *117'02. *255-07. Nr'P < L. =. Nor'P <, Df The following propositions (down to *255'108) merely re-state the above definitions. *255'1. F: a < v.. (2P, Q). Nu = Nor'P. v = Nr'Q. P less Q *255-101. F: u < Nr'Q. E. < Nor'Q *255'102. F: Nr'P <E v.. Nor'P < v *255'103. F: > v.. v < *255'104.:. v.:<. v. p, v e N0. t = smor"v *255'105. F:. J v.3: v <: v: V J. v., Z e NoO. = smor"v [*255104. (*255-05). *15544] *255'106. F: Nr'P < Nr'Q.. Nor'P < Nor'Q [*255-101-102] *255-107. F: Nr'P < Nr'Q. _. Nor'P Nor'Q *255'108. F:. Nr'P < Nr'Q. _: Nr'P < Nor'Q. v. Nr'P = Nr'Q. P e f [*255'107-104. *15516. *152-53] *255-11.: < yv. -. (gP, Q). P, QE. i=NOr'P. v=N r'Q. a! Rl'Q n Nr'P.,! Rl'P r Nr'Q [*255'1. *254-46] *255111. F: -> v.. (g[P, Q). P, Q e f. N = Nor'P. v = Nor'Q. g! RI'P n Nr'Q. -! RI'Q n Nr'P [*255-11-103] This proposition is exactly analogous to *117'1, except for the addition P, Q e Q. Hence except where this addition is relevant, the analogues of the propositions of *117 follow by analogous proofs. Such analogues will be given without proof in what follows, and will have the same decimal part as the corresponding propositions in *117. Where proofs are given, there are no analogues in *117, or else the method of proof is not analogous. *255'112. F:. p, v e NoO. D: u <i v. v. p = smor"v. v. v <u Dem. F. *255-1. 254-4. D F:. Hp.):,U < v I.V <. v (g{P, Q). P, Q e 1. = Nor'P. v= Nr'Q. P smorQ: [*155-4.*152'321] D: p, <z. v. v <,. v. (gP, Q)., = Nr'P. Nr'P= Nr'Q. Nr'Q = smor"v: [*155-16] 2: / <. v.. v <,. v. (aP, Q). u= Nor'P. Nor'P = Nr'Q. Nr'Q = smor"v: [*13'17]): < v. v <~. v.. =smor"v:. F. Prop

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62 SERIES. [PART V *255-113. F:.P,Qefl.):Nr'P<.Nr'Q.v.Nr'P=Nr'Q.v.Nr,'Q<.Nr'P Dem. F.*25 5 112 106.)I-:. Hp.: Nr'P < Nr'Q. v. N,)r'P = smor"'Nj'Q. v. Nr'Q < Nr'P: [*155-416])D: Nr'P < Nr'Q. v. Nr'P = Nr'Q. v. Nr'1Q < Nr,'P:.) F-. Prop [*255-112-104*105-103] *255-115. F:.P,Qd62.):Nr'P~<,Nr'Q.v.Nr'Q<.Nr'P: Nr1P~..Nr'Q.v.Nr'Q >Nr'P [*255-113-108] *255-12. F:.,a>v.=-:b'VeNOO: P e p Qcv.-)p, Q. g! RMYPA Nr'Q. r, g! R1'Q rNNr'P *255,121. F:.-P>v ii. 'ENO P ea~.)

p. (HQ) * QE iv. a! RL'P A Nr'Q. a! R1'1Q A Nr'P *255-13. F:Nr'P >Nr'Q. =.P, Q el. a! R1'1PANr'Q.- a! R1'Q nNr'1P *255-131. F: Nr'P > Nr'Q. =E. Nr'P >, Nr'Q. Nr'P + Nr'Q [*205513. *254-45] *255-14. F:,~z> v.- -.(aP, Q).-P, Q e 2-.L =NoL'P. v= Nor'Q. Nr'P > Nr'Q *255-141. F:,ut>v.=.,at>vq.tajsrnorI'v [*255-131-14] *255-15. V:- p> P. =. /.k, v E No0. a! s'RI"p n srnor"v. a! s'Rl"vAn smor"lp. *255-16. F: p.,vPENO. D: p > v.E =. stnor"I/i> v. =E. /.k> snorifv.=E. smorcc/..> smorl~v *255-17. F Nr'P >Nr'Q..Q less P.E.P, Qe!El.QeU'~Psm,. P, Q6 C 2. a! D'Ps A Nr'Q Demn. F.*255-13. *254-46.) F: Nr1P >Nr'Q. Q less P. (1) [*254-41] EP),Q E fI. QEU'PSM. (2) [*254412].P, Q e f. D'Ps nNr'Q (3) F.(1).(2). (3).) F. Prop *255-171. F:. PEf2. D: i< Nr'P. =. I-kNr"D'Ps- t'A Dern. F. *255-14.) F: Hp.)D: p. < Nr'P..(aQ), p= N,,r'Q. Nr'Q < Nr'P. [*255-17] E~(HQ).1. = Nor'Q. Q c f.! D(Ps NrcQ. [*1521].(HQ, -R). i= Nor'Q. QE 2. Q snor-. Re D'P,. [*152-35.*15516] E. (3R). =NrR. ReI2. Re D'Ps.~! [*2.53-18.*37 6] =-tE p.CNr"'D'P - t'A:.) F.Prop

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SECTION D] GREATER AND LESS AMONG ORDINAL NUMBERS B33 *255-172. F:. Pel.): 4 -< NrP.. (No)t. a C C'P. a! C'P n p'PYa. = Nr'P C a. a! j Dernt. F. *211703. *213-141.) 4 -I-: Qe DIP,:.).(aa). aCC'P. C p'P npfPCoa. Q=Pt a (1) ~.(I).*2555-M.1.: Hp. Cu < NrPP. D. (2[a) ca CcCP. f i CPnppP'Loa.tL =NrCPPCca.aI itL (2) F. *250-653. *25447.) 4 -F: Hp. a C C'P. H! C'P np'P1Ca.). PC a less P. [*255-17] Nr.PNr'P a <Nr'P (3) F. (2). (3).)IF. Prop *255-173. F:.PEfl.): Nr'Q < Nr'P C(~a) aCG'P.!C'P n p'PI "a. Q smor (P a) Dent. F. *255172-102. *155-22.D 4 -F:.Hp.): NrQ < Nr'P.r.((aa). a C C'P. a! C'P n p'P"a. N,rQ = NrCP a. [*152-35.*155-22] r.(aa). a C C'P. a! C'P n p'Pl"a. Q smor (P C a): D F. Prop *255-174. F: Nr'Q < Nr'P.. P e. Nr'Q E Nr"D'Ps Dem. F. *255-171-102-13.) F:. Nr'Q < Nr'P.: P E. N~r'Q e Nr"D'Ps - t1A: [*37-6.*155-22]: Pe fl: (aR). Re D'P. Nr'Q = Nr'R: [*155-16]:P e l: (qR). R e D'Ps. Nr'Q = Nr'R: [*37-6] Pe fl. Nr'Q e Nr"D'Ps:.)D F. Prop *255-175. F:Nr'Q, Nr'P..=PeI2.NrQeCNr"(D'Ps v t'P) [*255-174-108] *255-176. F:. fjP.):Nr,'Q,<Nr'P.=-.Pef.Nr'QeNr'C,'P [*213158. *255-175] *255-21. F: Nr'P< Nr'Q.. P, Q E ci.RI'P Nr'Q =A [*25451. *255-17] This proposition has no analogue in cardinals, because it depends upon *254-4. In cardinals, if Cl'laNc'/3=A, it does not follow that a! CI'/3A Nc'a, so that Nc'a may be neither less than, nor equal to, nor greater than Nc',3. *255211. F:. P, Qefl.): a! RI'PA Nr'"Q. a! R lQ Nr'P. =. Nr'P=-Nr'Q [*254-45] This proposition is the ordinal analogue of the Schrider-Bernstein theorem. If P and Q are series which may be not well-ordered, the proposition fails. Thus e.g. the series of rationals is like the series of proper fractions, which is

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64 SERIES [PART V a part of the series of rationals > 0 and 6 1, and this latter series is part of the series of rationals, but is not similar to the series of rationals, since it has a last term, which the series of rationals has not. *255-22. F: P, Q e f. a! Rl'P n Nr'Q. -. Nr'P > Nr'Q *255-221.:. Nr'P > Nr'Q. -: P, Q e 2: (R). R P. R

smor Q *255-222.: Q P. P, Q e. P. Nr'P, Nr'Q *255-23. F: Nr'P 3 Nr'Q. Nr'Q. Nr'P.. P, Q e Q. Nr'P = Nr'Q *255 24. F:, v. -. (gP, Q)., = NNr'Q. Nr'P 3 Nr'Q *255241. F:,u> v. =-.(atP, Q). = Nor'P. v= Nor'Q.P, Qe f. a! Rl'P Nr'Q *255-242. F:.,veNO. D: v..(P, Q).Pe,.Q v.!Rl'P nNr'Q *255'243. -:./ v. -: (sP, Q): P, Q e. = NOr'P. v = Nor'Q: (gR). R C P. R smor Q *255-244. F:./u, ve No.: / 3 v.. smor",/ v v. = smo. smor"v. -. smor" >smor"v a255*25.,:/u z v. v /,u...,, v e NoO. smor"l- = smor"v *255'27.: Nr'P < Nr'Q.. Nr'P < Nr'Q. Nr'P + Nr'Q *255-28. F: Nr'P > Nr'Q. -. Nr'P Nr'Q. - (Nr'Q > Nr'P)..P,Q e 2. - (Nr'Q. Nr'P) [*255'13-22-21] *255-281.: -> v.-.. (v /).-., v NoO.(v,) [*255.114] *255-29. F: Nr'P < Nr'Q.. Nr'P < Nr'Q. (Nr'Q < Nr'P). P, Qe f. (Nr'Q < Nr'P) [*255'115] *255-291. F: < v.-.. (v)..., v e NO. - (v e,) [*255'114] In the following proposition, we employ an abbreviation which is justified by its convenience, namely we put (tw). r e NO u t'i. Nr'P = Nr'Q + 4 instead of (ar). r e NO. Nr'P = Nr'Q -. v. Nr'P = Nr'Q 4 i. In virtue of *51'239, these two expressions would be equivalent if i had any independent meaning; but as i is only significant as an addendum, *51'239 cannot be applied. We will, however, adopt the following definitions: *255-298. (a[).eu vt'i.f(4u 4-).=:(-a). vex.f(/ 4-)).v.f(, + i) Df *255-299. We K l'i..f(Pl +).=: e K..f(/+ w):)f/(/+i) Df These definitions enable us to state many propositions, in which i occurs, as though i were an ordinal number.

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SECTION D] GREATER AND LESS AMONG ORDINAL NUMBERS 6 65 *255-3. F: NlP>N6.-PQ l(a)wcN l.rPN"~ Dem. F-. *255-175. *253-471.)3.:.r'PNr'Q = Pe f2: (aw).Nr'Q +w= Nr,'P. v. Nr'Q-tNrP [*251-132-26] =:P c (1: (aw). Nr'Q, w e NO. Nr,'Q+ wI = Nrl'P. v. Nr'Qei NO. Nr'Q - i-= Nrl'P: NN'-j-i = Nrl'P: [(*255-298)] —: P,Qe6f2:(Hw).wreNOvt'i.Nr'P=Nr'Qi-w:.:)F.Prop [*255-314] *255-32. F:.v, ~rENoO.:):v-iw.>v.E-.'r+Or Dem. F. *253 44.)F:HP-W+rOr:).-V-1i[V (1) F-.*255-31. D F:lHp. D. V+~'U (2 F. (3). (4). D F. Prop *255-321. F:. veNO.) VtOr..V +1i>v Dern. F. *253-45.:)I: HP -V+Or:).V0+ltV (1) F. *2a5531.:)F: Hp.:). v+1~i:, v (2) F. (1). (2). *255-141.3 F: Hp -Vt+Or.D).V+i-> V (3) F-.*255-141. D F: Hp. v +1 > v.D. v- i-tsmor""v [*161P2] D -V+ Or (4) F. (3). (4). D F. Prop Demn. F. *255-31. [*255-32-321] =R,&Wv e l N5 a) 1, v,.v.v+O.Po RI & WI III, 5

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66 SERIES [PART V *255-42. > (v W *255-431. [*255-43-114] *255-44. F: *255-441. ~.?~ [*255-44-114] *255-45. I 1~~>.)/> *255-482. F: 1% >v. v.>~VNOC. (>, *255-483. F -v..~,v,O (t *255-5. F:,ENO.2.fO:>, Demn. F. *255-31.)F:. 1u > Or. /ze N0(: wpf.?NO v t'1i [*180-61]:,eNO:.D F. Prop *255-51. F: ENoO- t'Or. A. > Or [*255-141P5. *153-15] *255-52. F:P c i~- fA.2Nr'P >2r Demn. F.*250-13.) F: Pe fI~- DA) E! B'P. [*56-11.*55-3] (a[y). (B'P) 4I y 2r A Rl'P. [*13-195]) a! 2rA RI'1P. [*255-22])N r'rP 2r (1) F.*255-22.)F: Nr'fP > 2r PE.a! 2r n Rl'P. F. (1). (2).)F. Prop *255-53. F: peNoO t'Or.=./Lt:> 2r [*255-52] *255'54. F:2. k2F=Orwvw,u~=2r Dern. F. *255'53. Transp.*255-281.) F: 2, >,a t = Or (1) F.

1).*255-105.)F.Prop

SECTION D] GREATER AND LESS AMONG ORDINAL NUMBERS 67 *255-55. F: F > 2r..~ e NoO - 'r - L2r Dem. F. *255'54. Transp. *255'281. F:.> 2r. -.u e NoO. P: Or. p += 2r: F. Prop *255-56. F: R e 12. Nr'P.>..NrQ.. NrR N'P > Nr'R + Nr'Q Dem. F.*255-3.) F:. Hp. D: P, Q,Re f: (ar). e NO u i. NrP = Nr'Q+ i: [*180-56] D: P, Q.,R e Q: (gs). a e NO v t'i. Nr'R +Nr'P-=(Nr'R + Nr'Q)+i: [*255-31.*251-26] D: Nr'R + Nr'P > NrR i- Nr'Q:. 2) F. Prop *255-561. F:eyeNoO.a> 83.D). y+ a > 7y+ [*25556] *255-562. F: Re Q2. Nr'P, Nr'Q. D. Nr'R + Nr'P > Nr'1R + Nr'Q Dem. F. *1803.) F: NrP = Nr'Q. D. Nr'R + Nr'P = Nr'R + Nr'Q (1) F.(1). 255-108-56.) F:. Hp.): Nr'R + Nr'P > Nr'R + Nr'Q. v. Nr'R + Nr'P = Nr'R + Nr'Q: [*255-108]): Nr'R + Nr'P > Nr'R + Nr'Q:.) F. Prop *255'563. F:e NO. a /3.). y a y [*255562] *255-564. F: P, Q,R e fQ. Nr'R + Nr'P = Nr'R + Nr'Q.. Nr'P = Nr'Q Dem. F. 25542.) F: Hp.. (Nr'R + Nr'P > Nr'R + Nr'Q). [*255-56.Transp] D. (Nr'P > Nr'Q) (1) Similarly F: Hp. D. (Nr'Q > Nr'P) (2) F.(1). (2). *255-113.. Prop This proposition establishes the uniqueness of subtraction from the end. Owing to the fact that ordinal addition is not commutative, we have to distinguish " subtraction from the end" from "subtraction from the beginning." They may be called terminal and initial subtraction respectively. Thus by the above proposition, terminal subtraction among ordinals is unique. This does not hold in general for initial subtraction among ordinals. *255-565. F: a,,3, e N0O. y a = y -/. D.a= smor",L [*255'564] The above proposition is still true if we put a = / instead of a = smor"/3 in the conclusion, but in that case it is only significant when a and /3 are of the same type, whereas in the above form it is free from this limitation. 5-2

68 SERIES [PART V *255-57. H:P,Qef~-t'A.).Qless(Px Q).Nr'Q<.Nr'Pk~Nr'Q Dem. F.-*250-13 D)t: Hp.). E!BRIP. (1) [*165-251]). Q smor Q 4, (B'P) (2) I-.(1).*166-1.)i-:Hp.:).Q4,(B'P)C-PxQ (3) F-.(1). *93-101.)I-: Hp. D. (ax). (B'P) Px (4) I-. *166-113.)1-:(B'P)Px.RECOQ4,(B'P).yEC'Q.).R(PxQ)(y4,w) (5) F:. Hp.)D: (2x, y): R eC'Q I(BP).)DR. R(P xQ) (y I x):y I xeG(Px Q) (6) Q 4, (B'P) smor Q. Q 4, (B'P) C P x Q. a! (J'(P x Q) rip'PxQ"rIC'Q 4, (B'fP). [*254-54])D. Q less (P x Q) (7) F-. (7). *255-17.DIF. Prop *255-571. -: a,I3,ENoO-tff~rD.)./< akf [*255-57] *255-572. -: P,QEfl-tfA. E!B'P.).P less (P xQ). Nr'P<.Nr'P*<Nr'Q Dem. -. *250'13.)IF: Hp.). E! B'Q.1 [*166-111] D. (B'Q) I; P CP xQ (2) F-.*151P64. (1). DH:Hp D.). (B'Q),P smor P (3) F..*202-511. D I-:. Hp.-): B'Pep'P"D'TP: [*166-111] D) x cD'P. ye JLQ.).D V(B'Q) 4, x} (P xQ) {y 4, (B'P)} (4) F. *202-511.) DF:. Hp.): B'Q e p'Q"UQ: [*166-111]::x= B'P. yEUGQ.)D. {(B'Q) I x} (P x Q) fy I (B'P)} (5) F. (4). (5).)- F:. Hp. D: xe CP. yEPGQ.XDGI(B,'Q) I x} (P xQ) {y I (B'P)}: [*150-22] D): Me C(B'Q) I;P -yeW GQ.-M (P x Q){fy I(B'P)}: [Hp.*33-24.*166-111]): (3N): NE C'(P x Q): Me C'(B'Q) 4,P. M. M (P xQ) N (6) F..(7). *255-17. DI-. Prop *255-573. F:.a,/3eNO-t'0r:(a[y).eeNO-t'0rut'i.a=y-~i:).a<-a,,f Dem. F.*204-483.)F:Hp.).(a[P,

Q).a=Nor'P./8=Nor'Q.a[!B'P (1) F. (1). *255-572.) DF. Prop

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SECTION D] GREATER AND LESS AMONG ORDINAL NUMBERS 6 69 *255-58. F:7ENO-L'07.a>P.:).a;<y>fl(ey Dem. F. *255-31.:) F. *184-35.:)F: a i- +.). a;(y = (03 y) -I-(*y (2) F. *184-16. D:H-+O D- <+O (3) F. *184-41. D F: Hp.caz=/3-i-).c;rky ==(,83*y-)-i-y. [*255-32] D. a;<y >/3*y (5) F. (1). (4). (5).:) F. Prop *255-581. F:Pcf2. E!B'P. QlessR.). P x Q less P x R. Nti'P*,<Nr'IQ < Nr'P*,~Nr'R Demn. F.*254.55.)F:Hp.).(2jS).Ssmor-Q.SC-R.a!C'Rnp'-R"C'S (1) F.*16611.:)F:SC-R.:).PxSC-PxR (2) F. *166-23.)DF: Ssmor Q D..P xSsmorP xQ (3) F. *202-524. *40-53.)DF:. Hp. zeGC'P. weG'S. yEcCIR np'R"C'S.)D: zP (B'P).V. z-=B'P WRY: F. (4). *166-111.)DF:. Hp.,y eG'R np'R"G'S.)D: F. (5). *10-28. D F:. Hp. a! C']?n p'R"C'S.): (aN): Ne C(P xR): Me C(P xS).Dm. M(P xR) N (6) F. (2). (,3). (6). D F:. Hp.- Ssmor Q. S C R. a! C']? np'R"C'IS.:): (P x 5) smor (P x Q).P x S C P x R. a! C'(P x 1?) rn p'P x R"C'I(P x 5): [*254-54] D. P xQless P xl (7) F. (1). (7).)DF: Hp. D. P xQ less P xR (8) F. (8). *255K17.:)F.Prop *255-582. F:.aeNoO:(a8).83eNO-t'0rVtfi-ax==3+i:/3<.y:). a * /3 <a X~y [*255-581.*204-483] *255-59. F:a,/,8, yeNO.7#0r,.a;<ry=/83(y.D.a~smorl"/3 Dem. [*255-112] D. a = srnor"/31: D F. Prop

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70 SERIES [PART V This proposition establishes the uniqueness of terminal division, i.e. division by an end-factor. Initial division (i.e. division by a beginningfactor) is only unique if the divisor is of the form -Ii. *k255-591. F:. a, P8 ry e N,O: (g8). 866 NO - t1O, w t~i. a =/i a*8=a; y:). fl= smor'1y [*255-582-112] *255-6. F: Nr'P > Nr'Q.. i-Nr'P i>i-Nr'Q Dem. F. *255-33.:.Hp.): (ar)., ueNO - t'"0. Nr'P Nr'Q-i- V V.. Nr'P +-Or. Nr'P = Nr'Q -: [*181-55]:: (,,4w).,ae NO - t'Or. i Nr"P = (i -/ NrcQ) a. v.. Nr11P + O,. i -j NrIP = (i -i Nr'&Q/ i [*255-33]): 1 - Nr'P > i- Nr'"Q:. I-F. Prop *255-601. F: Nr'P >Nr'Q.nQ. =-i-Nr'P> i-i-Nr'Q Dem. F. *255'6 QP *255-103. D P, Q F:Nr'P < Nr'Q. i —Nr'P < i -jNr'Q (1) F.(1). *255-108.) F: Nr'P Nr'fQ.:. i.-jNr'P j4- i Nr'Q (2) F.(2). Transp.*251-142.) F: i -iNr'P, i -/ Nr'QE NO. 0(i-Nr'P 1 i -/Nr'Q).). Nr'P, Nr'Q e NO. (NP,,,- NrQ) (3) F.(3).*255-281.)F: Nr'P > 4Nr'Q. D. Nr'P >Nr'Q (4) F. (4). *2556. D F. Prop *255-61. F: Q, Ref2. Nr'P=Nr 'Q-Nr'R.GT'R =-G'R.E!B'R.). Nr'P >i-I Nr'Q-i-i Dem. F. *253-57. F F: Hp. Nr'P - i Nr'Q 4 [4- Nr'R. [*255-32]). Nr'P i i > Nr'"Q- D:) F. Prop *255-62. F: Q,R e f~.Nr'P = Nr'Qi- Nr'R.Nr'Rt Or. (G (R, R= (1"R. E! BCR).). Nr'P > Nr'"Q Ii. Nr,'P 1> Nr' Qj 1 Dem. F. *253-571.) F: Hp.).Nr'P = Nr'Q-i Q jNr'R. [*255'32] D. Nr'P > Nr'Q 4- j. (1) [*255-321]. Nr'P - i Nr'Q -1Q (2) F.(1).(2). F. Prop

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SECTION D] GREATER AND LESS AMONG ORDINAL NUMBERS 7 71 *255-63. F:

Nr'P>Nr'Q.).Nr'P-~i->Nr'Q-j-1 Dem. F. *255'33.) F:. Hp.) (jR). Nr'1?+ 40, Nr'P = Nr'Q-jNr'R.v. Nr'Q + 0,. Nr'P = Nr'Q ij 1 [*255-62:321]: Nr'P - i- >Nr'Q - i-:.) F. Prop *255-64. F: Nr'P >Nr'Q.. Nr'P-jj> Nr'Q -iDem. F. *255-63-103.)F: Nr'P < Nr'Q.). Nr'P-I < Nr'Qi- 1 F. *181L31. D J- Nr'P-=Nr'Q. D.Nr'P —= Nr'Qj- (2) F. (1). (2). *255-113.) F: P, Q e fl., (Nr,'P > Nr'Q).). [*255-12]).(Nr'P-i j> Nr'Q -i-) (4) F. (5). *255-63.)FD Prop Dem. F. *255-53-31.) F:. Hp. w c NO -t'O,.. v = a — z): (Hp). p eNO u ff1. v=1 i-ip: [(*255-298)] D: v= /-t ~i - aV. V= /-j -li -I -j1. V. (Hp).p cNO- t'O,.v=1 I —1i -1P [*255-33] D: v> aiI(2) F.*255-45-321.) F: Hp. v > -i.:).v >,pa.F. (3). (4).:) F. Prop (4) The following propositions are concerned with the relations of ordinals to the corresponding cardinals, i.e. to the cardinals of the fields of well-ordered series having the given ordinals. If P is a well-ordered series whose ordinal is a, CGia = Nr'C'P, so that C"I'a is a cardinal whose members can be wellordered. Such cardinals have the property that of any two which are not equal, one must be the greater. If the cardinal number of one series is greater than that of another, so is the ordinal number; but the converse does not hold except for finite inumbers.

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72 SERIES [PART V *255-7. F. Nc"C'P" = C""NO [*152-7. (*251011)] *255-701. -. Nc"cC"n- t'rA=C"'(NO-i'A)=C"'NO-t'A [*255-7.*37-45] *255171. 1- P less Q) D. Nc'(C'P Nc'G'Q Dem. F 241Rc iN" F.*254-1. FClHp). C!RIQ n Nr'CP. [*I117-22] D. NcP'P P Nc,'C'Q: D F. Prop *2551711. F: Nr'P <.., NrQ.) D. NcPC'P <, NecC'Q [Proof as in *255,71, using *255,22] *255-72. F:a /3.).R D-Ccca <,C"/ Dem. F. *255'24.)I[*255-711] [*152-7] * Hp.:). (HP, Q). a = N~r'P. / = N,,r'Q. Nr'P s,- Nr'Q.D. (HP, Q). a = N~i'P.8 /= Nr'Q. Nc'C'P <-, Nc',C'Q.:). Cifia Ci1"f3:) F. Prop *255-73. F:.P,Qef~2.): Nc'G'P < Nc'C'Q.- v. NcP'P = Nc'G'Q. v. Nc'G'P > Nc'C'Q Dem. F.*255-711.) F: Hp. Nr,'P-<Nr'Q.). Nc'O,'P~<Nc'G'Q F. *255-71.:) F: Hp. Nr'Q < Nr'P.). Nc'G'Q < Nc'G'P F.(1).(2). *2 55 115.) F.Prop *255-74. F:. a,/3eC"'NO-tfA.):a~/3,.v.a>/3 (1) (2) Dem. F*255-701.) F: Hp.:). a,/36C""(NO - L'A). [*155'-34]:). (HP, Q). P, Q E fl a = C1"Njr'P,8/ = (i"Njr'Q. [*152-7]:). (HP, Q). P,QE2. a=Nc'C'CP./=N,,c'G'Q (1 F. *255173. *117-106'107 108.)D F:. P, Q e 12.): N,,cC'GP<, N,,c'C'Q. v. N,,c'C'P > NocP'IQ (2) F. (1). (2).) F. Prop *255175. F:P,Qe6i~.Nc'C'P<Nc'C'Q.).PlessQ Dem. F.- *1 17-291.) F: Hp. D. (Nc'G'Q -< Nc'IC'P). [*255-711.Transp]:).- (Nr'Q <- Nr'IP). [*255-29] D. Nr'P < Nr'Q. [*255-17] D. Pless Q: DF. Prop *255176. F: i,/3cNO. C9Ia< C"/3.).a< /3 [*2.5375.*152-7]

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*256. THE SERIES OF ORDINALS. Summary of *256. In the present number, we have to consider the series of ordinals in order of magnitude. Propositions on this subject deserve close attention, because it is in this connection that Burali-Forti's paradox* arises. This paradox, as we shall show in the present number, is avoided by the doctrine of types. But before discussing the paradox, it will be well to explain various propositions which raise no difficulty. For convenience of notation, we shall, in the present number, employ the letter M for the relation "<i". (This letter is chosen as the initial of " minor.") Thus "aM3 " means that a and /8 are ordinals of which a is less than 13. M'/3 will be the class of ordinals less than /3, M1',/ will be /3+1, and M1'/3, when it exists, will be such that either M1',8 - i = 3, or 3=2r. M1'/ =0,.. Thus (P'M, is the class of ordinals having immediate predecessors, and B'M, is the class of ordinals not having immediate predecessors. We have (*256-12) F:. aM/3.: a,, e NO: (g[y). 7 e NO- t'0r v ui. = - a y, that is, one ordinal is less than another when something not zero can be added to the first to make it equal to the second; *256-11. F: P e I D. M'Nr'P = Nr"D'Ps I.e. the numbers less than that of P are the numbers of the proper segments of P. Also, if P e Q, M M'Nr'P = Nor(Ps DiP,). Nor P D'Ps e I1 (*256'2-201), so that (*256'202) the series of ordinals less than that of P is similar to the series of the proper segments of P, i.e. to P (I'P (in virtue of *253'22). It follows (*256'22) that every section of M is well-ordered, and therefore that M is well-ordered (*256'3), i.e. that the ordinals in order of magnitude form a well-ordered series. * "Una questione sui numeri transfiniti," Rendiconti del circolo matematico di Palermo, Vol. xi. (1897).

 For the purposes of the present number, it is convenient to include 18 (cf. *153) in the series of ordinals; we therefore get N = M w Or Is 1V (i,/1) f '(M Dft [*256]. The effect of this definition is merely to insert 1, in the series M between r0 and 2r. We then have (*256-42) Nr'N= i - Nr'M. Now if Pet, P I'P (as we have just seen) is similar to a proper segment of M, so that if we omit to mention types we obtain -: P e f. D. Nr'P (I'P < Nr'M. Hence Nr'P, which is i - Nr'P ('P, is less than i +Nr'M (by *255-63), i.e. is less than N. Hence F: P e f. D. Nr'P < Nr'N. Nevertheless Ne Q2, so that it might seem as if Nr'N must be less than itself, which is impossible by *255'42. Hence we are led to Burali-Forti's paradox concerning the ordinal number of all ordinals. Burali-Forti's own statement of his paradox, which is somewhat different from the above, may be summarized as follows. Assuming a, BeNO. D:a</,. v. a= 3.v.a>3, (A), we shall have a e N0O..a <m a +i. But we also have a e NO.). a < Nr'N. Hence Nr'N < Nr'N i. Nr'N- i < Nr'N, which is impossible. The conclusion drawn by Burali-Forti is that the above proposition (A) is false. This, however, cannot be maintained in view of Cantor's proof, reproduced above (*255'112, depending on *254'4). The solution of the paradox must therefore be sought elsewhere. With regard to Burali-Forti's statement of the paradox, it is to be observed that "a < a - i" only holds if a! a+ i, i.e. if ()P). P e a. C'P V. This will always hold if a exists and is infinite, because then, if Pea, P I'P 4 B'P e a -i. But if a is finite, this method fails, since P (I'P B'Pea. Thus if the total number of entities in the universe (of any one type) is finite, "a < a+ i " fails when C("a = tV, which is just the crucial case for Burali-Forti's proof. Hence as it stands, his proof is only applicable if we assume the axiom of infinity; it might, therefore, be regarded as a reductio ad absurdum of the axiom of infinity, i.e. as showing that the total number of entities of any one type is finite. In order to make it plain that the paradox does not depend upon the axiom of infinity, we have above stated it in a form independent of this

SECTION D] THE SERIES OF ORDINALS 75 axiom. The paradox, stated simply, is as follows: The ordinal number of the series of ordinals from Or (including 1,) to any ordinal a is a 4i; hence a +i exists, and is therefore > a. But the ordinal a is similar to the segment of the series of ordinals consisting of the predecessors of a, and is therefore less than the ordinal number of all ordinals. Hence the ordinal number of all ordinals is greater than every ordinal, and therefore than itself, which is absurd; moreover, though the greatest of all ordinals, it can be increased by the addition of i, which is again absurd. In order to dispel the above paradox, it is only necessary to make the types explicit. In the proposition P e f. D. PlessN (B), upon which the paradox depends, the relation "less" is not homogeneous. N is of the same type as M, which is defined as Nr;less, where C'less =. Thus Nr'P e C'N. Thus N, as it occurs in (B), should really be N C t'Nor'P, i.e. N tt'tP, i.e. N(P, P), according to the definition *65'12. We have therefore *256-53. F: P e f.. P less N: t'Nor'P but this does not allow the inference N t'Nor'P less N V t'Nor'P, which is what would be required in order to elicit a paradox. The correct inference is, substituting for N t'Nor'P the equivalent form N (P, P), N(P, P) less N {N(P, P), N (P, P)J, or, more generally, *256 56. F. (N C X) less {N C (t'to'X)} Thus in higher types there are greater ordinals than any to be found in lower types. This fact is what gave rise to the paradox, as the corresponding fact in cardinals gave rise to the paradox of the greatest cardinal. *256-01. M=< Dft [*256] *256-02. N = M V 0r 1,s (t18,) f 'M Dft [*256] *256'1. F. M e Ser. CM C NoO Dem. F.*255-42. F. C J (1) F. *255-471. F. Me trans (2) F. 255-12. D. CM CNoO (3) F. (3). *255-112. *15543. 3 F. Me connex (4) F.(1). (2). (3). (4).:) F. Prop The above proposition assumes that M is homogeneous, since otherwise "C'M" is not significant. But M is significant even when it is not homogeneous. Thus the conditions of significance in the above proposition impose a limitation upon M which is not always imposed upon M.

76 SERIES [PART V *256-101. F: t! M. aC'm = NoO.Or= B'M: NO -tO, = I'M Dern. F-.*200-12. *256-1.)I-F.G'Mr'..'el F. *255-51. D F: pe NOO - COr.E OrMPA F.-(3). D)F. NoO -t'Or C P'M.Or (IEI'm -. (4). *2-6-1. D F.GI'1M C NOOLtOr F.(4).Q(5). (6). DIF. Prop The hypothesis ft! Af will fail in the lowest type for w significant, if the universe contains only one individual. Undex circumstances, ft! Af must hold. (1) (2) (3) (4) (5) (6) vhich M is rany other *256-102. Dem. F. (1). *33-24.)IF. Prop (1) *256-11. F:PceS2.). M'Nr'P=Nr'D'P, [*225-174] *256-12. F:.aM/3.n=: a,/3EN0,O: (a7c).ryeNO-t'Or.8a+~r.v.atOr./3==a-i [*255'33] *256-2. -: PCfD. D. Dem. ~ M (M*'Nr,'P) = Nr;P,. M ~ (M'Nr'P) = N~r;(Ps ~ D'6Pv) F.*256-1O1.:)F: Hp.PeO,.:).M~M*'Nr'P=A&.M~(M'Nr'P)=A& (1) I-.*213-3.)F:Hp.PE6,.:).NoirPs=A. Nor;(Ps4D'Ps,)=A (2) F.*256K11.*213-158.:)F:Hp. P,Or.). M*'Nr'P =Nr"'C'Ps (3) F.

(3). *255-17.)F:. Hp. P r- eor.): a {M (M*'Nr'P)1 /3.= (aQ, B). a= Nor'Q. 3= Nor'R. Q, R (C'P. Q lessRB. [*254-47].(HQ, R). a =N,,r'Q./3= Nor'B. QPsR. [*150-4] a a(N,,r;Ps)/ (4) Similarly F:.Hp.P,-e r0.:):a {M ~(M'Nr'P)} 8.=. a {Nr;(Ps ~D'Ps)} /3 (5) F. (1).(2). (4). (5). DF. Prop *256'201. F: P Efl.). N~rrD'Pe {M (M,'Nr'P)1smor (P, ~D'Ps)N,,r ~ GPsce M ~(M*'Nr'P)1 smor Ps [*253-461.*256-2]

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SE~CTION D] THE SERIES OF ORDINALS 77 *256-202. H: P e i~.). Nr'{M C (M'Nr'P)} = Nr'(Ps C D'Ps) = Nr'(P C U'P) [*256201. *253'22] *256-203. -: P E 1..). Nr'tM C (M*'Nr'P)1 = Nr'Ps [*256-201] *256-204. F:aN eO N f2-.). ij2Nr'(M I M'a) = a Dem. F. *255-10. *256202.) F:. P E fi. a = N-r'PP.): Nr'IM C M'aJ Nr'(P I Ga[P): [*204-46-272]): P - e6 2,. D. Nr'(M M'a) = Nr'P:. F. Prop *256-21. F:,c ENO. Pe,.)..M'a=Nr"D'Ps [*256-11] *256-211. I: Ez NO - f'O,. P e p). M*'L = Nr"C'PC [*213158. *25621] *256'22. F: eNO.).M M2'p~E&2 Dem. F. *256-203. F: Hp. P E p.). Nr'(M t M*'t) = Nr'Ps. [*t253-24] ~ M ~ M*"/-4 1 f (1) F. F: p + A R. D. M M M"Ik,6 f2~ (2) F.(2). *250'4. D)F. Prop *256'221. F:paENO.:).M~llfLeeI [*256-202] *2563. F. M Cn [*256-221. *250-7] *256-31. F:ft!M.).2,r=2m= Ml'r Dem. 4- 4 F. *255-51-53. F: Hp.). M'Or = t'2r v M'2r 4 -[*205-196.*2561]). 2, = min(MMOr [*206-42.*201P63] = M"'Or [*250-42.*256101] = 2m:) F. Prop We shall have, for every finite v, V, = VM, where V, will be defined as the ordinal corresponding to v, i.e. as f, r, CCCV. (This is a single ordinal when v is finite; otherwise, it is the sum of a class of ordinals.) This subject will be considered in the next section.

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78 SERIES [PART V *256-32. F:.aMi38.=:a,fleNO:a#O,.i3=a-j-i v.a —(,Or so32r Dem. F. *255-65.:) I- a E No0 -t "Or. M'a = t'(a +1w M'(a —) [*205-196]).a A-yi= inm'M'a. [*206-42.*201P63])a-j-iMica (1) F-.(1).*256 31.) F. Prop *256-4. F. ISC',e NO Den?, F. *153-36. DF-: R 18.).G'RE I. [*200-12.*250-12]).1? C f (1) F-. (1). *251-122.:)IF: a eNO.D. a n18= A (2) F-.(2). *153-34. DI F.Prop *256-41. F. N=MwjOrjl,V1 j(t'1s)TI(1PM [(*256-02)] *256'411. F:.aN/3.Ea =0Or so3E tc18U (1M -V. a=18.3c(1I1J~~a,$E1PM~AI/3[*256-41] *256-412. F: M =A.)..N=Or4,1s. Ne2r [*256-41] *256-413. F: M=Or42r.).N=Or4,1sVJ~r42r~Jlr42r.NEi1#2r [*256-41. *161P211] *256-414. F:P(IM, —,e1)N =0Or lst-~M (1J'M Dem. F. *204-46. *256-101.) F: Hp. Aj! M.:).N= Or<+FM ~ (P 'J Or 4I, isW'(tclg) T C'(M1I (M) [*161-101] = Or 4, 18 WJ(Or V ti) T~ C'(M ~ QWI') iV M ~ "1IM [*160-1] =Or 4,1s4~M G('M (1) F.(1).*25 6412.:)F.Prop *256-42. F:ft!M.:).Nr'N=i1+Nr'M Dem. F. *25 6414.) F: Hp. CU'M,,e 1.).Nr'N=- 21.i- Nr'(M~ (PGM) [*181P57] =i i-i Nr'(M ~ 1VM) [*204-46] =ijNr'M(1 F. (1). *256-413.:)F. Prop *256-43. F:NeQ.-t'A& [*256-412-42]

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SECTION D] THE SERIES OF ORDINALS 79 *256-44. F:.PEf2.):.P %I'PlessM.=E. PlessN. [!M Dem. F.*255517'601.) F:. Hp -): P C G'P less M. NrIIP C GIP < Nr"M (1) F.*256-412-42.: F: P = A.). P less N (2) F.*205551. D F:. P = A.): P CI'IP less M. M (3) F. (2). (3). F. P = Ai. D: P G CIP less MC._. P less N. ft! M 4 F.*200-35. *25 551. D F:KI 'P e 1.I: P 71F'P less M. =-.!A M (5) F. *256'42.:) F:. Hp.U'1P e 1.!1M.:I). P less N (6) F..(5).(6)). D -:. Hp.. II'Pe I1.:PPt CTCPless M.=-.:k!i M. PPle. — N (7) F. *204I46.) F:. Hp. t! P. U'IP e 1.): i i-Nr'P U6'P = Nr'P: RIA D~ P C (I Y less M._. NrY < i Nr6M. [*256-101-42] E.Nr'P < Nr'N. A! M (8) F. (4). (7). (8).) F. Prop We now make use of the above propositions to show that every wellordered relation P of the type we start from is less than N, where N is to hold between ordinals of the type to which N~r'P belongs. This proposition embodies what Burali-Forti's paradox becomes when account is taken of types. *256-5. F: f! M. P f2. O. N,r;(Ps t D'P,) e D'(M t'N~r'P), Dem. F. *2562. *253-13 F: Hp.). N,r;(Ps ~ D'Ps) D'f (1) F.(1). *150022.) F: Hp.:). N,r"D'PsCtt'G'MCM. [*213-141] N. Nr'P e t0'G'Ms. [*63-531). t'cCGM = t'Nor'P (2) F. (1). (2).) F. Prop *256-51. F: P f.. Nr;(Ps C D'Ps) smor P ~ I'P [*253-463] *256-52. F:! Al. P ef.). P C PP less Me t'Nor'P [*256551. *254-182] *256-53. F: Pe fl. D. P less Nt'Ntr'P Dem. F. *256-44-52. D F: Hp. f! M.. P less N C t'Nor'P (1) F.*256-102.)F:Hp.M=A.).P=A. [*256-43]. P less N (2) F. (1). (2). DF. Prop *256-54. F: Pef&.). Nr (P)'(N t'Nor'P)= A Dem. F.*256'53.) F:.Hp.)::Qct'P.:). '-.'IQsmorN~t'N,,r'P}: [*152-11]): t'P A Nr'(N C t'N~r'P) = A: [(*65,04)]:): Nr (P)'(N C t'Nr'P) = A:.) F. Prop

80 SERIES [PART V *256-55. F: Pe l. -. Nr (P)'(N t'TNorP) = Nr (P)'(N C t't'P) = Nr (P)'IN (P, P)} = A Dem. F. *155-12.. Pe Nor'P. [*63-105]. P e to'Nor'P. [*63-53]) -. t't'P = t'Nor'P (1) F. (1).) h. Nr (P)'(N t'Nor'P)=Nr (P)'(N t't'P) (2) [(*65-12)] = Nr (P)'{N (P, P)} (3) F. (2). (3). *256-54. 2) F. Prop *256-56. -. (N C X) less {N C (toot')} Dem. F. *256 43 53. F.(N C X) less {N C (t'Nor'N X)} (1) F. *155'12.) F. N X e Nor'N: X. [*63-105]) F. N V X to'Nor'N X. [*63-53] F F. t't'N \ x= t'Nor'N C X (2) F. *64-16. 3F. NY e t'(to'X to'X). [(*64'01)] D F. N e too'x (3). (2). (3).:) F. ttoo'\X = t'Nor'N C X (4) F. (1). (4).)h F. Prop When types are neglected, the above proposition appears as N less N, which is impossible, and embodies Burali-Forti's paradox. In the form proved above, however, the paradox has disappeared, and we have instead the proposition that in higher types longer series are possible than in lower ones.

*257. THE TRANSFINITE ANCESTRAL RELATION. Summary of *257. In this number, we are concerned with an extension of the notions of R* and Rpo. This extension requires two relations, R and Q. It is most easily explained by first defining the " transfinite posterity " of a term x with respect to R and Q; this class is an extension of R*'x. This class is generated as follows. Let us suppose, to aid the imagination, that Q is more or less serial in character, and that R is a many-one relation contained in Q. Then the transfinite posterity of x with respect to R and Q is generated as follows: Starting from x, we travel down the posterity of x with respect to R (i.e. R*'x) as long as we can; if the whole class R*'x has a limit with respect to Q, we begin again with this limit, which is to be included in the transfinite posterity of x with respect to R and Q; if the limit is y, we travel 4 - down R*'y, and include the limit of this class with respect to Q, and so on, as long as we still have either terms belonging to D'R or classes belonging to (ItQ. The whole of the terms so obtainable constitute the transfinite posterity of x with respect to R and Q, which we will denote* by (R*Q)'x. In order to obtain a symbolic definition of this class, let us call a class a "transfinitely hereditary" when not only R"a C a, as in the ordinary hereditary class, but also if we take any existent sub-class, of a en C'Q, if p has a limit with respect to Q, that limit is to be a member of a-. Thus a is to be such that the R-successor of any member of a belongs to a and the Q-limit of any existent sub-class of a- n C'Q belongs to a (so long as these exist). That is, R"a C a- and p C a.!a C'Q.3. I. ' C o-. Using the notion of the derivative of a class with respect to Q, introduced in *216, the condition p C a-.! C n C'Q. t. Qlt C a reduces to SQ' C a, in virtue of *216*1. Hence a is transfinitely hereditary with respect to R and Q if R" u SQ'(C a-. * This meaning for R*Q has no connection with the meaning temporarily assigned to this symbol in *95.. &W. III. 6

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82 SERIES [PART V We may now define the transfinite posterity of x with respect to R and Q as all members of C'Q which belong to every transfinitely hereditary class to which x belongs, i.e. we put (R*Q)'x = C'Q n x e o. R"'a u Q'o C o.,. y e o} Df. Then the analogue of R* is 2 {[ye(R*Q)'x}. This relation, however, is less important than the analogue of Ro limited to the posterity of x. This analogue, assuming Q to be transitive, will be Q ((R*Q)x. For this we introduce the two notations Qx and Q (R, x), the latter being more convenient when either R or x is replaced by a more complicated expression. Thus we put QRx= Q (R, x) = Q C (R*Q)'x Df. If Q is a well-ordered series and R= Q, QR is merely the series Q 4 — 4 -beginning with x, and (R*Q)'x = Q*x = aQ'x u tix if xe C'Q. Thus in this case, if x = B'Q, Qpx = Q. But the importance of QRx is in cases where Q is not completely serial, but becomes so when limited to (R*Q)'x. In these cases, Q will, in applications, almost always be logical inclusion combined with diversity, or the converse of this; i.e. it will be either a/3(a C 3. a /3) or MN(M N. M N), or the converse of one of these. In the case of a/3 (a C /3. a = 3), we have ItQ = s r (-Ci'maxQ). tlQ =p r (- ('minQ), as will be proved in *258. In the present number, we are concerned in proving that, under certain circumstances, QR e Q1. The proof proceeds on the lines of Zermelo's second proof* of his theorem that if a selection exists from all the existent subclasses of a given class, then the given class can be well-ordered. Before proceeding to treat of this subject, however, it is necessary to prove some elementary properties of (R*Q)'x. These are given in the propositions preceding *257'2. We have *257-11.: x e a. R"a -v SQa C a. D. (R*Q)'x Ca Thus in order to prove that (R*Q)'x is contained in a class o, we have

University of Michigan Historical Math Collection: Principia mathematica, by Alfred North Whitehead ... and Bertrand Russell.

to prove (1) that x belongs to ar, (2) that the R-successors of members of a are members of a, i.e. that ao is hereditary with respect to R, (3) that the derivative of a- with respect to Q is contained in ar, i.e. that if p is any existent sub-class of a n C'Q which has a Q-limit, this limit is a member of a. * " Neuer Beweis fur die M6glichkeit einer Wohlordnung," Math. Annalen, LXV. p. 107 (1907). His first proof, which was somewhat more complicated, was published in Math. Annalen, LIX. p. 514 (1904).

SECTION D] THE TRANSFINITE ANCESTRAL RELATION 83 *257-111.. (R*Q)'x CC'Q *257-12. F: x C'Q. =. x (R Q)'x *257-123. F: R C Q. D. R"(R*Q)' C (R*Q)'x I. e. if RC Q,(R*Q)'x is hereditary with respect to R. The hypothesis R C Q is required for most of the properties of (R*Q)'x. *257'125. F: RCQ.xeC'Q..R*' C (R*Q)'x Thus if x e C'Q, the R-posterity of x is contained in (R*Q)'x. *25713. F: p C (R*Q)'x. a! p..ItQ' C (R*Q)'x *257'14.: R C Q. D. (R*Q)'x C Q*' Thus (R*Q)'x is wholly contained in the Q-posterity of x. The following propositions (*257-2 —36) are concerned in proving QRxe2, with a suitable hypothesis. This hypothesis is Q e Rl'J n trans. R e RQ n Cls - 1. ltQ r Cl ex'(R*Q)'x e 1 - Cls. We assume, to begin with, only part of this hypothesis, namely, Q e Rl'J n trans. R e RL'Q n Cls -) 1. Thus to prove Qx e Ser, we only have to prove QR e connex, i.e. y e (R*Q)'x. *. (R*Q)x C Q'y, or, what comes to the same thing, (R*Q)'x Cp'Q"(RQ)Yx. Let us put a-' = (R*Q)'x n p'Q"(R*Q)'x. Then any member of o-1 may be called a " connected term," because it is connected by Q or Q with every other term of (R*Q)'x. (A connected relation is then a relation whose field consists entirely of connected terms.) We wish to prove that cr- is a transfinitely hereditary class, and therefore equal to (R*Q)'x. We do this, not directly, but by combining a-1 with another class a2 defined as follows. Consider those members z of (R*Q)'x which are such that their successors in QX consist of R'z and its successors in QR,, i.e. put 4- +- - 4 T = (R*Q)'x n z tQR'z = (Q)'R' Z. It will be observed that, even when Q is transitive, Q* and (QRx)* are still 4 — v useful. In this case, (QRX)* = QRX I C QA1, so that (Qp)*,'Rz consists of R'z and its successors in QR. We then consider the class a-2 consisting of those terms y whose predecessors are all members of T, i.e. we put 4- < — W a-Y = (R*Q)'x A {zQy. z (R Q)x.. Q z = (Qa,,)*'R'z}. Finally we put - = ao- n 0-2, i.e. —) 4- --- v a- = (R *Q)w n p'Q"(R*Q)'x n, {zQy. z e (I Q)'x.,.dQRxZ = (QRx)*'RZj} 6-2

84 SERIES [PART V The reason for this process is that it is easier to prove that a is a transfinitely hereditary class than it is to prove this directly for a-; and the result follows immediately for ao when it has been proved for a. We have then to prove R"a C a. SQ'/ C -. The first step is to prove y e a-. QR 'Y = QRX R y u l'R'y. This is proved by transfinite induction, by showing that —) v4 -Q*'y u Q.'R'y is a transfinitely hereditary class, whence the result, because, by hypothesis, --—) 4 -

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(R*Q)'x = (QRX)*'y QRX'y. The proof that Q*'y v Q*'R'y is a transfinitely hereditary class is as follows. If z e QI'R'y, 'z e Q'Ry. If z = y, R'z = R'y. — +4- 4 -- % v If z e Q.RXy, then since by the hypothesis QR'Z = (Qx)*R'Iz, we have -'- 4 --- v v - y e (QR)*RIz, i.e. R'z e Q'y. Hence z e (R*Q)'x n (Q*'y v Q*'R'y).. R'z e Q*'Y v Q*'R'y. We have next to prove -+ 4 — v - - 4 —v tI C (R*Q)'x n (Q*'y v Q*'R'y). a! p.. t It C Q*'y v Q-'R'y. 4 — v - v If a! F t Q*'R'y, then ItQ'p C Q*'R'y. - 4 -4 -4 If p C Q*'y. y e p, then y e maxQ'p, and ItQp = A. -0 4 -If p C Q'y, we have y ep'Q"p, whence w ItQ,. D. (yQw), whence, since y, by hypothesis, is a connected term, wQ*y. - - 4- - v - 4 — v Hence in any case ItQo' C Q*'y v Q*'R'y. Hence Q'y v Q*'R'y is hereditary, and therefore contains (R*Q)'x; and hence - 4 ---- v --- -- v QR' y =) (QR)'y. (Q)'y = QRx'R'Y This shows that R'y is a member of ao. For by hypothesis this holds of all predecessors of y, and we have now shown (1) that it also holds of y, (2) that y is the only predecessor of R'y which does not precede y. This is the first step towards proving that a is transfinitely hereditary. It follows immediately, from what has now been proved, that if y e oa, R'y (if it exists) is a connected term. For by hypothesis (R*Q)'x C Q,'y v Q'y, whence, by what we have just proved, — Rv x-R - v (R*Q),ax C Q'R'y QC'RCy,

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 whence R'y is a connected term. Hence R'y e a. Hence R"o- C o. It remains to prove 8Q- C a-. v +_- <- v Just as R"aCa was proved by proving Q'y =Q'R'y, so 8sQ'a C is proved by proving p'Q"i C Q*cItQNit provided,u C o. a!.! maxQ',u; v -- and this is proved by showing that Q"/L v Q*,"ItQo' is a transfinitely hereditary class. To show that Q"pf v Q*"ItQ'bt is a transfinitely hereditary class if, C Ca.!,. ~ E! maxQ'pt, we observe that by hypothesis +-_ -- 4 --- v, z E QR c t.. QRxZ = (QRX)*'R'z. D! Ih n (QR)*'R'z. Hence R'z e (QRx)*"/P; and hence, since by hypothesis pa C Q"/L, R'e QRX"P. Hence Ri{(Q*R)'x Q"/L} C (Q*R)'x n Q'r. Also obviously R"Q*"ItQ'pu C Q*"ItQ',F. Hence putting p = (Q*R)'x A (Q'" v Q*ItIV), we have RI"p Cp. We have now to prove 3Q'p C p, -+ -) i.e. a C p a a. -! maxQ'a.. ItQ'a C p. If a C Q,/p, it is obvious (since,p is composed entirely of connected terms) that seqQ'a C Q"t v ItQ'p.,v - On the other hand, if a! a n Q*m"ItQ'L, then a n Q"/p, if it exists, does not v - affect the value of the limit of a, which is the limit of a n Q*"'It,'Q, which is v - obviously contained in Q*"ItQ'P. Hence SQ'p C u. Hence, is transfinitely hereditary, and we have,/ C oa. g! p^..a! maxQ',p.). (R*Q)'x C Q" t v Q*"ItQ'P. At this point it is necessary to assume ItQ r Cl ex'(R*Q)'x e 1 -- Cls. This being assumed, we have, by what has just been proved, fC (a.!,. a! ItQp. 3. (R*Q)'x C Q"/t v Q'ItQ'. 3.* (R)4- Cl- u -Ql). (R*Q)'x C Q'lto'% v Q*'ItQ'Ft.

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 Hence ItQ',/ is a connected term. Hence Q'a C p'Q" (R Q)'x. We only require further — I, v+_~4 4_ --- %,u C a.!.! ItQ'. D: zQ ItQ'. z e (R*Q)'x. Z. QR'Z = (QR)*'R'z. Now by what we have just proved, zQItQ'z. =. ze

Q1"p; and by the definition of a, since, C (, we have 4 --- v Z e QC * * QRX'z = (Qi1X)*'R'z. Hence we arrive at 8Q'acr-. Since we have already proved R"-C a, it follows that a is hereditary, and (R*Q)'x C a, i.e. 4- 4 --- --- y e(RQ)'x:,: y ep'Q(R Q)'x: zQy -I. QR.'Z =Z(Q)* R'z, 4- --- v i.e. QRx e connex: z e D'QR. z. QXZ = (Qlix)*'R'Z. Hence Q e Ser. Hence also the immediate successor of every term z in D'QRx is R'z, so that D'QRx C D'R. (QR), = R C (R*Q)'x. To show that QR e f2, we observe that every class contained in D'Qi? has a sequent, namely seq (Qx)'A = x, -^ v a C D'QR. a! maxQa.. seq (Qlz)'a = R'maxQ'a, a C D'QRX..! a. a! maxQa.. seq (Qx)'a = ItQ'a, whence a C D'QX.). E! seq (Qx)'a, which shows that QRx e. The first derivative of Qx is SQ'(Q*R)'x, and its last term, if any, is L'{(Q*R)x - D'R}, i.e. ItQ'{(Q*R)'x n D'R}. The hypothesis required for QR e Qf is the same as for Q1, e Ser, namely, Q Rl'J n trans. Re R'Q n Cls -- 1. ItQ r Cl ex'(R Q)'xe I - Cls. In order that QRX may not be null, we require further x e D'R. The next set of propositions (*257-5 —56) are designed to prove that, subject to the above hypothesis together with x e D'R, Q,, is the only value of P fulfilling the following conditions: (1) P is transitive. (2) C'P is contained in (R*Q)'x. (3) If z is any member of D'P, R'z is its immediate successor. (4) If a is any existent class contained in C'P and having no maximum, ItQ'a is its P-limit.

 This proposition is essential for what may be called CC transfinite inductive definitions," i.e. definitions of a series by defining the successor of every term, and the successor of every class having no maximum. The following illustration may make this clear. Suppose, R is a manyone relation of classes to individuals; suppose we start with some class a, and proceed to a v tfR'a, a v tRaV t'R'(a v t/ R'a), and so on. At the end of this series we put its sum, i.e. its limit with respect to the relation (C A~ J); let the sum be fi. We then proceed with fi v t'1?'/3, and so on, as long as possible. The series ends with a sum which is not a member of D'R, if there is such a sum. It is evident that the series is uniquely determined by the above method of generation; the above-mentioned propositions give symbolic expression to the process expressed in words by " and so on, as long as possible." *257-01. (R*Q)'x=C'Qt-5^{xe-. R'J-uQa-CO-.:)c.yeo-I Df *257-02. QRx = Q (1?' x) = Q (1?*Q)'x Df *257-11. F:: yec(1*Q)'x.:yEcC'Q: xeo. Ro-vV C. *257-102. I-::yc(R*Q)'x.=n:.yeC'Q:. WCa.R"a C a-: p C a-. aj!p~ (i'^Q. ~.maxQ'Ft.)3. seqQ',uC a:.e a[*257-101. *207-1] *257-11. Fxe a-. R"o- v(Qo- Coa-.:). (R*Q)'x Co- [*257-1] Almost all proofs of propositions concerning (R*Q)'x use this proposition. *257-111. F.(R*Q)'xCC'Q [*257-1] *257-12. F:xeC`Q =.x.e(R*Q)'w [*257-1] *257-121. F:?C-Q-ye(R*Q)'x.D.R'yC(R*Q)'x Dem. F.*257-1.)F:.Hp.yRz.): Xea-.R"o-v3Q'o-Co-.)..y~ea-:yRz.zeC'Q: [*37-1])z 6 'Q: xe a -R"o- Ca-. Q'a Ca-ez Ca: [*257-1] [*257~~~1] D: zec(-R*Q)'x:. D F. Prop

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88. SERIES [PART V *257-122. F:1RC Q. pC (R*Q)'cx.. R"FsC (R*Q)'x [*257-121] *257-123. F: R CQ.). R"cl(R*Q)'rxC(R*Q)'x [*257-122] *257-124. F-:1?C Q.). R*", (R*Q),'x C(R*Q)'w [*257,123] *257-125. F:1? C Q.- x e C'Q.). R*'x C (R* Q)'fx [*257-121'1 *257-126. F:RCQ.xeD'R.,-.(xRx).). (R*Q)'wx-.,E~ 1 [*257'125] *257-13. F:,tC (R*Q)'x. a! /.. I.tQc/tC (R*Q)'x Dem. F.*257-101.*10-1.*221.)F:I C (R*Q)'Ix.):. xE6 o-. Rlco C a-: v C a-. a! v ni CJQ.),.. tQcv C a: D C aF. (1). Fact.) D F:: Hp.)D:. x a -. Rcco- C a-: v C a -.a! v ^ (1Q.), tQcV C a-: D. P C O-. a! IL F:. v C a-. a1! v Ai G'Q.),. ltQc) Cao-::): Hp.j 1.C a -.yltQ/L.) y eaF. (2). (3).: F:: Hp. y ltQF4.k):. x ca-. Rcco- C a-: v C a-. a1! v v' (iYQ. D,. tQcv C a-: D. y caF. (4). *10-11i21. *257-101. D F: Hp~. yj ltQ/. D. ye (R*Q)'1t: D F. Prop *257-13i. F. 8Q'(R*Q)'x C(R*Q)'wx [*257-13. *216-1] *257-132. F:K CCl ex,'(R*Q)'Ix.:). tQCK C(R* Q)'Ix [*257-13] *257-14. F:RC-Q.D.(R*Q)'xCQ*'x Dem. F. *90-163.) F: Hp. D. R"cQ*'w C Q*'x F. *206-15.) F:u,C Q*'x. z ltQU. a! P.). z ep'Q"cl. a!,a'., a C Q*'x. [*40-61.*90-163]).z e Q" P aQ"c C Q*'x. [*22-46] z. z e X F. (1). (2). *257-11.)D F: Hp. x~ e C'Q. D. (R*Q)'w C Q*'x F. *37-261-29. *60,33. (*216-01). F: Hp.).D Rl"(- C'Q) = A. 8Q"& Cc'Q) = A F. (4). *257-11. D F: Hp. xw, e C'Q. D. (R*Q)'x C - C'Q. [*257-111] 2.(R* Q)'Ix= A F. (3). (5). DF. Prop *257-141. F: RC-Q.2. R"C'Q u8Q'G'QC UQ [*216-111.*37-201-16] I I 1.24] 1 (1) (2) (3) (4) (1) (2) (3) (4) (5)

SECTION D] THE TRANSFINITE ANCESTRAL RELATION 899 *257-142. F:RC Q. xeU'Q.. (1R*Q)'x=gtxea-.RoR"a-u Q'o-CCa -.:)u.ye a Dem. F.*257 141.)F:Hp.). txea-.R"a -v Q'a-C a -..)oyeoa-1 C'Q (1) H (1).*2571. 3.F.Prop *257-15. 1: y (R*Q)'x. z E (R*Q)'y.). z E (R*Q Q)'x Dem. F. *A257~1.) I-:. RCa- v Bq'o-C a-.) ~: x eOa.). a-: Y E a-.)~. z e a-: [Syll]):xea.).z~a (1) I-.(I). *257K1.)I-. Prop *257-16. I-: xe C'Q- D'R.). (R*Q)'x= t'x Dem. -. -*257-12. DI-: Hp. D. xe (R*Q)'x (1) -. *37-261-29. D F: Hp. D. Rt'LLx = A (2) F.*205-18. F: Hp.,-,i! maxQ' x.). xQx. [*206-42]).seqQt'x = A (3) F.(3).E*216011.)F:Hp.. Q'cx = A (4).(2).(4).( ) F: Hp..RQ'xv Q'xCC x. [*2c5711]]). (R*Q&)'x C ffx: (5) F.- (1). (5-).) F. Prop We now begin the proof (completed in *257-34) that under certain circumstances QRx 1-. We first prove that the class a introduced in *257-2 is transfinitely hereditary, and this requires as a preliminary the proof that -— p f-~- ' if ye a, the class (QRx)*'Y V(QRx)*'R'y is transfinitely hereditary. This preliminary is provided by *257-2-21. The hypothesis of *257-2 is not all used in *257-2, but is introduced because it is required in the set of propositions of which this is the first. *257-2. F:. Q e RL'J n trans. R e Rl'I"Q Cls -1. 4- 4 — a- = (* Q)'x n pi'Q"(R * Q)"x n 9 IZQRXY D Qx=,*r D =: tL- -)I Id J + -- % Y * a Z C (QR.)*'Y V (QRx) *'R',Y z e D'R - R'z C (QIx)*'Y " (QRx)*'R' Y Demn. F.90-163 - *37-62. *257-123.) 'S1, 4- 'S 'S -- ' F~:. R C Q. E! R'Iz - D: z e (QB,)*cRC. y. R'z e (Q~x)*"Rcy (1) F. *30-37. D F: E! R'z.z=y. R.z= R(y (2) F. *201-18.*91-52.*32-182. D F: Hp. z RxQi'y. D. - QR.Z = (Q&x)*'RJ z '.Y C QRx'Z [*13-13])-Y6 (QRx)*'R' z. [*32-182] D. R1?z e (QRx)*'y (3) F.(1).(2). (3). *71-161.:)F. Prop

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90 SERIES [PART V *257-21. I-:Hp *257-2.y E a.,C (QRX)*'y V(Q.Rx)*'R'y. H! p.). ItQ'pO C Q*yv Q*'-R'y Dem. F. *201-14-15 *206-134. F. *205-38.) F: Hp.p LC Q*'y. y e4. D.ye maxQ'/zt [*207-11]).tQ'1z= A (2) F. *40-5 5. *206 143.) -4+ 4- '-4 -F:,U C Q'y. W tQLU.)y ep'Q"L. w 6EQ,"p'Q"11. F. *257-13.)DF:. Hp (3). Hp. D: yQw -v. wQ*y: [(3)]:): wQ*y (4) F. (1). (2). (4).) F. Prop *257-211. F: Hp*257-2-. y E.:). (I?*Q)'x C (Q~x)*'Y U~ (QRx)*'1?'y Dem. F. *25 7-14.) F: Hp. x e(Q.Rx)*'y (1) F. (1). *257-221-11.. F. Prop 4- 4_- %.-' — 4 -4+ %'*257-22. F: Hp*257-2. Y -.)-Q. = (QRx)*'R'Y. (QP.x)*'Y =QRx'-RY Dern. F.*257-211.:) F: lip.).(QRx)*'R',Y = (R* Q)'X - (QRX)*'Y [Hp] =QRx'?I (1) Similarly F: Hp.. (QRX)*'6y = Q~ ~Y(2) F. (1). (2).) F. Prop It is to be understood that (QRx)*'R'y = A if E! ly. *257-23. F:Hp *257,2. D.R"o- C a Dem. F. *257-22.:)F:. Hp.y e a-rD'1R.): zQR'y.)Z. QRx'Z =(Q.x)*'R'z (1) F.*257-22-211.)F: Hp.y E o- D'R.).(R*Q)'Ix =QRx'R'yv~'(QRx)*'R',y (2) F. (1).(2).:)F:Hp.yEaAD,'R.:).R'yEa::)F.Prop The above proposition gives the first stage in the proof that a- is transfinitely hereditary. The second stage, similarly, requires as a preliminary the proof that if p is an existent sub-class of a- having no maximum, then '5x -4XVCV is a transfinitely hereditary class. This proof is provided by *257-24-241242.

SECTION D] THE TRANSFINITE ANCESTRAL RELATION 91 *257-24. FH*5-.kaal.-! aQ/-)RQxtCIxi Dern. F. *91-52. *201-181.:) F: Hp ze Q~x',.). Qpj.z = (QRx)*'RCZ. [*,3746.*1 3-1 2]).a!(QRx)*'-R'z n F' [*37-46] C?Z E (QRx)* "/-J(1 F. *205-123.)F: Hp.).'C QRX"F' (2) *257-241. F: Hp *257-24.:). R"{IQ1~x"'~ v (QRX)*"IItQ'F'} CQRxP~kC(QYRx*1CtQIL Dern. F.*90-164.)F:?C-Q.:).R"(QRix)*"itQ'FLC(QRx) *"IIt'Q ', (1) F.(1).*2572."4.) F. Prop *257-242. F: Hp *257-24. p = ~xc v (QPX)*" ItQ'F' a C p. a! a.*a maxQ'a.) tQ~a C p Dern. F.*206-15.)F: llp.2!Ltnp'Qa.wltQa.:).H!F'-Q'w (1) F.*201-521.)F:Hp.,tCo-.).ct-Q'IwCQ*'w (2) F. (1).(2).)F:Hp(1).).a!F'AQ*'w (3) F. *205-123.) F: Hp.). jk C Q"F' (4) F.(3). (4).) F: Hp(l).D).wEQRX"F' (5 F.*206-24-:)F:Hp./.'CQc"a.aCQI"F'.).It.Qa=ItQ/Lk (6) F. *206-15.) F P: Hp. a! a Ai (QR-x)*"ItQ'IIk.:). It~'a C (Qr~x)*CcItQ',L (7) F. (5). (6). (7). D F. Prop *257-243. F: Hp *257-24.).(R*Q)'x QBxl"lu v'p~"QFx' [*40-53. *205-123] *257-25. F: Hp *257-24.:). (R* Q)'x =QRc"b' V (R)tl~t Dem. F. *257 -242) F:Hp.). 8Q'IQRx"F'Uv (QRX)*"ItQ'F4ICQIx"'tV(Q.Rx)*"ItQ'Ft (1) F. (1). *257-241.)F. Prop *257-251. F: Hp*257-24.).(QRx)*"itQ'F'=P'QRx"F', Dem. F- *257-25-243.:) F: Hp.) D Q~x"F v (QRx)*"CLtQ'F' = Q~"x vc/UP'Q~x"ll. [*200-53. *24A481])-("tQ't-pQ,"F:F. Prop

92 SERIES [PART V *257-252. -: HP *257-24. 2!p'QBX1"1,.). Qp,"",=p'QRj'ItQ'1t.! ItQ/hu Dem. F.*257 25 1.*37 29.) F: H p. g~!ItQ',a (1) -+ -4 %a J [*200-53.*40-62]:).pQRx"ItQ'1ak C (R*Q)'X - (QRx)*"'tQ'11 [*2-7-251] C (R*Q)'x -P'Q~x"1a [llp. *10O57.*257-24~3] C Qp' (2) F.*201-.51. *4067.:)F:Hp:). QRx'ALt C P'(Q jj'ItQ'P

(3) F. (1). (2). (3).) F. Prop In order to complete the proof that a- is a hereditary class, we have to introduce the additional hypothesis ItQ r CII ex'I(R *Q)'x E 1 -), Cis. With the help of this hypothesis, the last stage of the proof isprovided by the following proposition. *257-26. F:HP *257-2.ItQr~Cl ex'(1?*Q)'x E1 — Cis.).8~Q' C aDem. F *257-251-2-52.) F.Hp. p~ C a. tt!p. g! ItQ/AO.: (R* Q)'X = Qp" ItQ"/L V (Q~x)*'ItQ')UA. QPx~tQ 'p= Q.RI'p: [Hp]) tQc" EP'Q "(R *Q)':X YQRxl1tQ'P.-:) if-.~ = (Q~x)*'R',Y [Hp]): ItQ/4 e a::.) F. Prop *257-261. F Hp *257-26. -).(1? *Q)'x= a-[*257-1123-26] *257-27. F:Q eRIfJ ntraDS.?e RI'Q iCls - 1. ItQ r Cl ex'(R* Q)'x e -), Cis.).D QRxe Ser. QR= (R I Q*) (R*Q)'w Dem. F. *257-261.) F: Hp. D.- (R*Q)'x CP'CQ"C(1*Q)'x rn ^ {zQRxy. -). Q~x'z =(Q1)*'R'z} (1) F. (1). D F:: Hp.)D:. QRxe connex:. z eD'Q,,,,.:),: zQRxw. zR I(Qj~)*w:. [*5-32.*4-71. *257-121]):. Q.Rxeconnex zQ. Rxw. ZeD'QR., zR Q*w. WE C'QRx:. [*36K13. *257K121]):. QpRxe connex. Q.Rx (R I Q*) ~ (?* Q)'x:::) F. Prop We have thus proved that QR, is a series. No additional hypothesis is required to prove that it is well-ordered, as we shall now show. -+ 4 -*257-28. F: Hp *257-27. C (R*Q)'x. p t. maxQL'L = A. g!p&11p p'Q~x",L= (QRX)*"IItQ',L. QRX11P = P'QRX"CItQ',-t [*257-251 -27]

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 *257-281. -: Hp *257-28. E!ItQ/A.:). P'QRX"IL" = (QRX)*'CItQcaL - R 6/= QRX,jtQ'tk [*257-28] *257-29. F: Hp*257-27.WE D'R.:). C'QR = (R*Q)Ix.B,'QRX = x Dem. F.*257-27-126. *202-55.) F:Hp.).C ~= (R*Q)'x (1) F.*2 57 14. DF:Hp.:).(R*Q)'x-icXcQB.jX (2) F. (1).-(2). DIF. Prop *257-291. F: Hp*257-27.xWJD'R.D. QR=A [*257-16. *200-35] *257-3. F:Hp*257-27.).D = D'?ri (R* Q)'x Dern. F.*257-27. DF:. Hp. ye(R*Q)'x. D: g! Q'y H! Q*'R'y. [*257-141] =E! R'y:. DF.Prop *257-31. F:Hp *257-27. pC (R*Q)'x. a! tk. a!maxQ L. a! p'Q.Rx"/"U. D se(Rx)',a= ItQ/A. [*257-28] *257-32. F: Hp *257-27./tC (R*Q)'x. a! rnaxQ /L. [PQR,'{-) seq (Q~,'xt R'rniax (QR.)'IL Dern. F. *257-3. D F: Hp./.zC D'R. [*257-27.Tranisp]).Q*'max (QRp)'t = Q'R'max (QR~x)'bt F. Prop *257-33. F:HP *2'7-27. C (R*Q)w. a!z. a! p'Qp.",.). E!seq (Q~x)'F [*257-31-32] The above proposition together with *257'27 shows that QR,, is wellordered, in virtue of *250-123. *257-34. F: Hp *25 727.). QRx c 2 Dem. F. *2-57291)F:Hp. x cD'R. D.Q~x efl 1 F.-*257-29. *206-14. F:Hp. x eD'R.) se(Ip'A = x (2) F.(2). *257-33.) F:. Hp. w eD']?.:): p C (R*Q)'x. a!p'QRx"j"p.:),. E! seq (QR,,)'U: [*257-29.*206-131] D: a! P'QRx"(P Al GQRx).). E! seq (QRX,)',ut [*250-123.*25'7-27]:): QR,612 (3) F. 1). (3). DF. Prop

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 *257-35. F: Hp *257-27.. R C (R*Q))'w = (Qx),. B R (R*Q)'xI c 1 -* Derm. F.*257-32.)F:. Hp.)H:py D'yQx. D. seq (Q&R)'t'y=R'y (1) F.(1). *206-43. *204'7.)F. Prop *257-36. F: Hp*25-2Z.xED'R.). O&r Q~= (R*Q)'x. U' Q = (BR*Q)'x - t'x. BQRx = x. B'Q,,x= (R*Q)'x- D'R [*257-29-3] The following propositions are concerned in showing that a relation P which satisfies the hypothesis of *257-5 is identical with QRx, thus showing that this hypothesis is sufficient to determine P. *257-5. F: Hp*257-27. PEtrans. C'P C (R*Q)'x. P_- P2 = R t (1* Q)'x. Itp r Cl ex'(R1?*Q)' = ItQ r Cl ex'(R*Q)'x.). P C J. C' = (R*Q)xr The above hypothesis is not all necessary for the present proposition, but it is necessary for the series of propositions of which this is the first. Dem. F. *37-4L.) F:. Hp.): D'(P - P2)= R"I(R*Q)'x A (R*Q)'w [*25736] = (R*Q)'x n D'R (1) F. *3214. DF: Hp.). Itp'{(R*Q)x' A nD'R = ItQ'I(R1*Q)'xr D'1R} [*257-36] - (R*Q)'x - D'R (2) F. (1). (2).: F: Hp.). (1?*Q)'x C C'P. [Hp]).(R*Q)'x = CP (3) F. (3).)F: Hp.):xeD'P.).xP_ — P2(R't). [*34-5.Transp]. (XPX) (4) F. (3). (4).) F. Prop 4 - *257-51. F: Hp *257-5.). CP = P'x Dem. F. *257 123. *90-16.)F: Hp.)." R'P*'X C P*'X (1) 4- 4 -F. *90-13.) F: Hp.)D. RtQ"Cl ex'P*'x = Itp""Cl ex'P*'x. [*90O163. *40-61] D. ItQ'"C1 ex'P*'x C P*'x (2) F. (). () F Hp.:..(R*Q)'x C CP*"x (3) F. (3). *257'5.) F. Prop In order to prove P= = we first prove P e f2. The proof proceeds as for Q,, but in some points it is easier. It is merely outlined below, as it closely resembles the proof for Q~. *257-52. F: Hp *257-5. f* 4- 4 -= CP n p'P"PC'P (zPy. D,. P'z = P*'1?z).). R"Ca Co

SECTION D] THE TRANSFINITE ANCESTRAL RELATION 9 a Dem. F. *345. Transp. *201 18) F:. P,= R (R*Q)'"x yc p'P"G'"P.): zI> (R'y). D. —, (yPz): zP~yy. D. zP (R~y): [Hp] D: ZP (RI~y).. zP~y 1R' As in *257-2-21, using Itp r Cl ex'(R*Q)'x = ItQ r Cl ex'(R* Q)'x, we prove F: Hp. y D a i D'R. p = P*Iy v P*'R'y. D. R"'p C p. 8Q/p C p. D..(R*Q)6xlC P,yv u PRRy (2) (1). (2). D F: Hp. y e a n DCRR. D. P~y = P*"R'Ry (3) F. (1). (3). F: Hp. y E a n D'R D. R'y E a-: D F. Prop *257-521. F:Hp*257.52./ttCa -.!p.c-=!irnaxp'1.). (R* Q)'x = P(Clk V P* CCRAU [Proof as in *257-25, by similar stages] 4- 4- ' *257-53. F:. Hp *257-5.)D: P e Ser: ze D'P. D, P'z = P*'R'z [Proof as in *257-27] *257-54. F:Hp*257-5.).Pef~ [Proofas in *257-34] -+ — 4 W *257-55. F: Hp *257-5 - a- = y (P'y = QIjy).). Ra- CaDent. F.*257-53. F F: Hp. ye CG'P. D. P'R'y= CG'P-Pj*'-R' y 4 -[*257-53] = G'P - P'y [*257 53] -P'ylffy (1) F. M. D F-: Hp. y D.. Plf.R(Y = QR.' V t" [*257-22] = Q&x'R'y: D F - Prop *257-551. F: Hp *257 55. D. 8Q'o C a Dem. F.*257-53 -) F: Hp.P Ca [a!, a. z = ItQIIP. D. P'z = {(R*Q)'x nAl v P"tL [Hp] = t(R*Q)'x n Aj V &R.' cp [*257-27] -Q= xz: D F. Prop *257-56. F: Hp*257-5.) P= Q, Dem. F.*25 7'51,54. D F Hp. D. P'x =A. [*257,36] D. P'x=Q = x (1) F. (1). *257551 551F:.Hp. D: y e C'P.)Y.IP'y = QR/yy:.)D F. Prop This proves that the conditions in the hypothesis of *257-5 are sufficient to determine P.

*258. ZERMELO'S THEOREM. Summary of *258. In this number, we shall first show the applicability of the propositions of *257 to the case where the Q of that number is replaced by logical inclusion combined with diversity, i.e. by any one of the four relations: ai (a C 3. a +), a8/3 (C a. a +.,), MN(MCN.M:N), MN(NCM.M +N). If we put Q= a/t (a C/. a +$), and if K is any class of classes, then s'K is the maximum of Kc with respect to Q if sKc e K, and the sequent of K with respect to Q if s'8cK e C K (*258'1'11); similarly p'Ic is the minimum of Ki if pKc e c and the precedent of / if p'K(.ee K (*258'101'111). Hence every class of classes has a unique maximum or a unique sequent with respect to Q, and every class of classes has a unique minimum or a unique precedent (*258*12); we have, moreover, tQ = s r (- (mraxQ). tlQ =p [(- ('minQ) (*258'13'131). Hence ltQ, tlQ e 1 -- Cls (*258'14), and Q and Q therefore satisfy the most exacting part of the hypothesis of*257'27. Also Q and Q are Dedekindian relations (*258'14). (They are not series, because they are not connected.) An exactly similar argument applies to MN(M N. M: N). Hence if Q is any one of the above four relations, and if R is a many-one contained in Q, it follows from *257'34 that Q with its field limited to the transfinite posterity of any term is a well-ordered series. If we take Q = a,(aC/.a =+3), and take any initial term a, our series proceeds to continually larger classes, proceeding to the limit by taking the logical sum, i.e. if K is any existent sub-class of the posterity of a, s'K = limaxQ'K = limax (QRa)'K (*258'21'22), where QpB has the meaning defined in *257. This process stops with s'{D'R A (R*Q)'x} if D'R n (R*Q)'x has no maximum; otherwise, it stops with the R-successor of this maximum, which is maxQ'fC'R n (R*Q)'x}. If, on the other hand, we take Q to be the converse of the above, we proceed to continually smaller classes, and the limit of any set of classes K having no last term is p'c. In this case, if, starting from a, every existent sub-class of a belongs to D'R, the process of diminution cannot stop short of A. This is

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SECTION D] ZERMELO S THEOREM 97 the process applied in Zermelo's theorem. We have the: e a class ju, assumed to be not a unit class, and a selective relation S for existent sub-classes of,u, i.e. a relation S for which S e a'Cl ex'p. Then our relation R is the relation of a to a- t'S'a, i.e. the relation of an existent sub-class of u to the class resulting from taking away its S-representative. Thus QR, is a well-ordered series, which starts from p and ends with A. Omitting the final A, S selects a representative from every member of the field of QRL, and the series of these representatives, i.e. S;QRL,, is similar to QR, with the final A omitted. Moreover every member of pJ occurs among these representatives, for, if x be any member of pL, let Ic be the class of those members of C'QR,P of which x is a member. (There are such classes, because p e C'QRx and x e P.) Then x ep'c, and by what was said earlier, pKc is a member of C'QR,. Hence, by the definition of Kc, p'eceK, and therefore pK = maxQ'/c. But no class smaller than p'c can belong to K, and therefore p'c - t'S'p'c is not a member of K, and therefore x is not a member of p'c - t'S'p'c. Hence x= S'p'c, and therefore x occurs among the representatives of members of C'QrR, which was to be proved. (The above is an abbreviated rendering of the symbolic proof given below in *258'301.) Hence the field of S;QR, is p, and therefore there is a well-ordered series having p for its field, provided ea'Cl exi'p is not null (*258-32). This is Zermelo's theorem. The converse of Zermelo's theorem has been already proved (*250'51). Hence the assumption that a selection can be made from all the existent sub-classes of, is equivalent to the assumption that p can be well-ordered or is a unit class, i.e.

*258-36. F: p, e C"Q v 1. E. [! cA'C1 exi' Hence also, by *88'33, the multiplicative axiom is equivalent to the assumption that all classes except unit classes can be well-ordered, i.e. *258-37.: Mult ax. -. C"fQ v 1 = Cls Hence also, in virtue of *255'73, the multiplicative axiom implies that of any two unequal existent cardinals one must be the greater, i.e. *258'39. F:: Mult ax. D:., v e NoC.:/ <v,. v.,u > *258-1. a:.Q = & (C. a +/3).:s'IKe c s..s= maxQ'Kc Dem..* 205'101. D ~:: Hp. D:. y maxQf'.=-: e K: ae Kc. Da. (y Ca. 7 y a): [Transp]:: 7 e K: a e. a ry. a (7 C a) (1) F. (1). *10-1.) f:: Hp. s'c e tK.:. y maxQK. - E K: a7 E IK a 7Y. Da-(yC a): SC K 7.. (7Y C sKc): [*4013] -: y e:a e.a + y). Da (y Ca): s' = y: [Transp. *40'13]: y e. S'K =: [Hp] E:SC =7 r::. Prop R & W. III. 7

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98 SERIES [PART V *258-101. 1-: Hp *258-1. p'Kc E IC.). " = minQ /K [Proof as in *258-1] *258-11. F: Hp*258-1. s'#c E K.). seqQ K = S'K Dem. F. *40'53.:) F: Hp.).p'QK-I^(caE K.:),. aC y. a+ 4y) [Hp.*40-151.*10-29] = (s'/C C y) (1) F.*40-1. *22-4246.) I-F. S"K =p'r (S'K C Iy) (2) I.(2).- *258-101.:) F liHp..s ScK minQry (S'K C fy) [(1) = seq Q'K: F. Prop *258-111. F:Hp *258'1. P'KC E —.').KprecQ'iK ==P'KC [Proof as in *258-11] *258-12. F:. HP*2581I.):E!maxQ'tc. v.E!seqQK: E! niinQ'c. v. E!precQKI/ [*258-1101-11-111] *258-13. F: Hp *258-1. tQ =S (-IGmaxQ) Dern. F. *258-1. Transp.) F:Hp. g! naxQ"K. D. S'K I E K. [*258-11]). ItQKIC= SK) F. Prop *258-131. F:Hp *258-1..tlQ =p [~(- GminQ) [Proof as in *258-13] *258-14. F:Hp *2581l.). Q, Q eDed. ItQ, tlQ E1 -+ Cls [*258-1213-131] *258-2. F: Hp *258-1. Re RI'Q Cls ->1.).QJ lEl Dem. F.-*25-8-14.) F:Hp..Hp *257-27(1 F. (1). *257-34. DF. Prop *258-201. F: Q=&a,3(I3Cca.a+I3). BERI1'QnCls —+1.:). QR,,Ecf2 [Proof as in *258-2] *258-202. F:Q =k&I(M CN. Mt +N). ReR1'Q ^Cls 1.). QRBXEf2 *258-203. F:Q=AN(NCGM.MtN).RER1'QnCls —+1.).QRX6f2 *258-21. F: Hp *2 58-2. K C (R *Q)'Ia. S"K = limaxQ'K Dem. F. *258-13.:) F: Hp., a! maxQ"K?. 8 SK = ltQ'K(1 F.*258-2.)F:.Hp.2[!maxQ'K.):(2 [ry):rYEK:aE/,c.)a.aCy: [*40'151] S):sKc~: [*258-1] S'K = maxQ'K (2) F.(I).(2).) F. Prop *258,211. F: Hp *258&201. K cC (R*Q)'a.).'K = litaxQ'Kc

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SECTION D] SECTION D] ZERMELO'S THEOREM 9 99 *258-22. 1-:Hp*2.582.a eD'R. K C(R*Q)'a. a!K.). "K = irnax (QRa,)'K Dem. F-. *258-21.:) I: Hp. S"IK EN1 K.).S"K = tQ'K./ [*257-13]).S"K E(R*Q)'la. [*210-233]).S"K = limax (QRa,)'K: F. Prop *258-221. F: Hp *258&201.a ED'R. KC (1*Q)'a.). p'i= (R*Q)'K *258-23. F:Hp *258&2.a ED'R.). Qpl eDed. s(R*Q)'a =B'IQRGL [*258-2,22. *250-23. *205-121] *258-231. F: Hp *2 58-201. a E D'IR.) QBp,,, Ded.p'(R *Q)'a =BQR *258-24. F: Hp *258-2.) (?* Q)'fa = /8 (a E a- R I"a- C a-. sl"Cl ex'o-r C a-.:)". /3E a-) Dem. F. *258-113. *257-1.)D F. *25 7 123.:) F: Hp.). R"',(R *Q)'la C (R*Q)'a (2) F.*257-12.)F:Hp.:).aE(R*Q)'a (4) F. (2) - (3). (4).) F.(1).(5).) F.Prop *258-241. F:Hp *258-201.). (R* Q)'a = j(a ca-. R%"a"-C a-..p"CI ex'a- C a-. 0,. 6 a-) *258-242. F:Hp *258-202.:). (R*Q)'X =Y(Xea-.R"o-C a-.~"'C1lex'o-C a-.,. Yea-) *258-

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243. F: Hp *258-203.)D. (R*Q)'X= Y(XEa-. R"o- Ca-.P"Clex'a-Co-.),.. Yea-) *258-3. F aQ=(/C a.at/3). Se ca'Clex'lfu R = a&8 (a e Cl ex'",a. / = a - ~'IS'a).) QBR, E. SQB,,smor QRB, - 'A Dem. F.*80-14.:)F:Hp.:).RCQ.ReCls-+1.D'R=Clex',u.C'R=Cl,', (1) F. (1).*258-201. D)F: Hp.D. QB,,E f2 (2) F.*257-35. D)F: Hp.:).R ~C'QB, el1- + I F.*80-14.)F: Hp.:).PC[S= Clex' I (5) F. (3). (4). (5). D F: Hp. D. S~;QR,,smor QB, VI-A) (6) F. (2).(6). F. Prop 7-2

100 SERIES 100 SERIES ~~~~~~~~~~~[PART V *258-301. Dem. I-: Hp *258-3. x EuA. I = C7'QR, A "X.). X= S'J'K F. *257-36. DF:H.:, "B [Hp] D!/ F-. (1). *258&241. D F-: Hp. D. pl'ice (R*Q)'At. [*257-36]:). P', e "Q~ F. *40-1. DI-: Hp.:). xep'ic F. (2). (3). F:- Hp.:).p"KEK. [*258-101] D. pl'c= maxQK/ F.(4). F p.(" "pi)-eK [*257-121.Hp] D. x -'E (p'/C - t'S'p'c) F. (3). (5). F-: Hp. D.xe t'S'p'lc: DF. Prop (1) (2) (3) (4) (5) *258-31. F:Hp *258-3. P I..G'SQQp-A Dem. F. *80-14.:) I-: Hp.). PS = Cl ex'"at. [*150-36.*257-14] S).,,= S;QR, CI(tA). C'QB,, (t'A) C PIS. [*202-54.*257-125])C'IS;QR,. -S"l(C'QR -t'/A) (1) F. *83-21.) F: HP) S"G'"QR, C (2) F. *258&241-301.. F:Hp. x p. D. xe S"{j(R*Q)'1t - t'A}. [*257-36] D. x e Sl(C'QB,,~- t'A) (3) F. (1). (4).:) F. Prop *258-32. F 4C xp "~ [*258'3-31] This is Zermelo's theorem. *258-321. F:Hp*258-3.,8QB~a.).S'I38.ea Dem. F. *250-242.:) F:. Hp.):a= (Q~p)1'fP - V. (Qpg) 1'/3QBIcEa [*257-35.Hp]): a C,3 - ffIS'j3:.)D F. Prop *258-33. F: Hp*258-3.pr —e1 c.P=S;Q,.:).S= minp rCl ex' Dem. F. *80-14. FHpaC.!.)Slaa F. *258-321.:)F: Hp (1).xE a.). r(Hl)./QB,La.x: [*150-4.Hp]:). -o(xPS'a) F. (1). (2). *205 1.:) F: Hp (1).:). S&a minp a. [*258-3]:). S'a = minp'a:) F. Prop = S',3. (1) (2)

SECTION D] ZERMELO'IS THEOREM 101 *258-34.H.rE1: SE eca'Cl ex"'. (2[P). P eI. C'P = uS= minp rCl ex'/.t [*250,5 *205833] *258-35. I-,JC'' ~ 4C x1t[*200-12. *250-51.*258&32] *258-36. F E vc C112 v I e4'CTl ex'",a [*258-35. *60-37. *83-901] *258-37. F:Mult ax. =-. Cll2v 1 = Cis [*258&36. *88-33] *258-38. F:. Mult ax.)D Nc'la < Nc'13. v. Nc'a = Nc',3. v. Ncla > Nc'j3 [*255-73. *258-37. *117Z54'55] *258-39. F::Multax.):.1wkveNC.):/.k~<v.v.it>v [*258&38]

*259. INDUCTIVELY DEFINED CORRELATIONS. Summary of *259. In the theory of well-ordered relations, we often have occasion to define a relation (which is generally of the nature of a correlation) by the following process: Given a relation S, let W'S be a relation (generally a couple) which is a function of S. Let us put A w'S= S W'S. Then, starting from A, we form the series A, A w'A, A w'A w'A, etc., each of which contains all its predecessors. We proceed to the limit by taking the sum of all these relations, i.e. s'(A w)*'A; we then proceed to v 4 ---A w'h'(A w)

*'A, and so on, as long as possible. The sum of all the relations so obtained is a function of W, and is often important. As an example, we may consider the correlation of two well-ordered series P, Q, which is dealt with in *259'2 —25 below. In this case, we put hA W = XT {X = seqp'D'T, seqQ'C'T}. Hence W'A = A w'A = B'P 4 B'Q = p 4 1Q, A w'A 'A = lp 1Q 2p 2Q, and so on. Proceeding in this fashion, we can continue until one at least of the two series P, Q is exhausted. We thus obtain a new proof that, of any two well-ordered series, one must be similar to a section of the other. For convenience of notation, let us put temporarily A=ST(S(T.StT) Dft. We then have A e Rl'J n trans. A we Rl'A n Cls -- 1. which is part of the hypothesis of *257-27 and following propositions. The rest of this hypothesis follows by analogy from *258'14. We now put WA= S'(A *A)'A Df. Then WA correlates the whole of P with part or the whole of Q, or vice versa. This is proved in *259'25, below.

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SECTION D] INDUCTIVELY DEFINED CORRELATIONS 103 For other values of W, we get other results, often of a useful kind; for example we shall have occasion to use the methods of this number in *273, which deals with series similar to the series of rationals. The present number gives, first, some elementary properties of (A wA)'A and WA for a general relation W, concerning which we only assume that W'S is never contained in S, i.e. W A () = A (except in *259121'13, where we also assume W e 1 -) Cls). We then proceed to deal specially with the case where W = XT {X = seqp'D'T, seqQ''PT} as explained above. h h *259-01. A =ST(SCT.S T) Dft [*259] *259'02. Aw= ST(T=S' W'S) Dft [*259] *259'03. WA= s'(Aw*A)'A Df In the following propositions, which result from those of *258, it is essential to have A A. For this we require that W'S, when it exists, shall not be contained in S. It will be observed that, according to the above definition, A, =ST(SCT). Hence instead of using " C" as a relation, which is notationally awkward, we shall use A*. Thus the condition we wish to impose upon W is that we are never to have (W'S)A*S. This is insured by W t A= A, which accordingly appears as hypothesis in the following propositions. *259'1.: A e Rl'J ^ trans. ItA e 1 - Cls: W iA A '. D. AWeRl'A n Cls - 1. A (Aw, A)e f Dem. As in *258-14, F. ltA e I - Cls (1) F. *201'18. D F:. Hp. D: MWS. D. (M C S) (2) F. (2). (*259-02). D:. Hp.: SA wT.. SC T. S +T. [(*259'01)]. SAT (3) I. (1). (3). *258-202. D F. Prop In the following proposition, the notation A (A w, A) is that defined in *257'02, adopted because A w cannot conveniently be used as a suffix. *259-11. h:E! W'A. WAA*=A.D. WA = B'Cnv'A (Aw, A). s''Cl'(A,*A)'A C (A w*A)'A Dem. F. *258-242. *259-1. D F: Hp. X C (AwA)'A. D. 'X e(A v*A)'A (1) F. (1). DF:Hp.). WAe (A w*A)'A (2) F. *4113. D F: Hp. T e (A* A)'A - t' WA. (. TA WA (3) F. (1). (2). (3). D F. Prop

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104 SERIES [PART V *259-111. F:. W AA* =A.S, T E(Aw*A)IA.): S CT. v..T CS [*259-1. *257,36] *259-12. F: SE D'A w. =. E! W'S [(*259-02)] *259-121. F: Wel-

University of Michigan Historical Math Collection: Principia mathematica, by Alfred North Whitehead ... and Bertrand Russell.

+)Cls.:).D,'Aw=W[,W [*259-12] *259-122. F: WA A*= A.xWAy.xk=(A w* A)'A ^T'I —;(xTy)}.:).x(W'A'X),y Dem. F. *259-11.)F: Hp.:). 41XE (A w*A)'A. (1) [Hp]:). 'xe x (2) F. (1). (2). *257-3.) F: Hp. D. eI D'A w. F. (2). (4). DF HpD. x hX. x(wc,)y [(*259-02)] D. x (W'Ih'X) y: D F. Prop *259-13. F: WA A*=A WeI-* Cls.) D. WA= 9'W"c(A v*A)'A Dem. F. *259K122.:) F: Hp.)D. WA C 9,'W"1,(Aw* A)'A (1) F. *257-123. DF: Hp. D. ~W"l(Aw*A)'lA CWA (2) F.(I). (2).)F..Prop *259-14. F:. WAA*=A&:SE(AW*A),'An1i-~Clsrt71'W.)s. W'SE I-+ Cls. GF 'SAW'S =A:)D. WA El -Cls Dem. F.*71-24. (*259'02).) F:. Hp. Se(Aw*A)'A.n 1-+Cls.).Aw'(SE(A W*A)'Anl- +~*Cls (1) F. *259-111.)DF:. Hp.S, T E(A W*A)'A.3: S CT. v. TGCS (2) F. (1). (5). *258-242. D F: Hp.:). (A w*A)'A Cl1 -> Cis. [*259-11] D. W`AEI —. Cls: DF. Prop *259-141. F:. W AA* = A:S e(Aw*A)'A rCls -+l1ri ('W. Ds. W'SEcCls — *1.DR'SAD ' W'S =A:)D. WAE Cls-+1I [Proof as in *259 14]

SECTION D] SECTIN D] INDUCTIVELY DEFINED CORRELATIONS10 105 *259'15. F:. WAA*=A:SE(Aw*A),'Anl1-)IrilG'W.:)S. W'ScI-+ 1..D'Sv n DIW'S= A. WGIS nQ'W'~= A:) TVAe [*259-14-141] The following proposition is a lemma for *273-23. *259-16. F:.-WAA*=A:Te(Aw*A)'AnP(IW.P~D'T=T;Q.DT Dem. P D'fWA= WA; Q: Te(Aw*A)'A.:)T.-P ~D'T=T;Q I-. *259-111. F:. Hp-X C (AW* A)"A.): F. (1).:) F:. Hp..XC(A w*A)IA: TE6X.:) ~D'T= T;Q:) [*259-111].(aS, T). S, TE X-x(S IQ IT) y [*1501] =~~~~~~ x {"); Q} y (2) F. (2). *258&242.)F: Hp. Te(A w*A)'IA.:).P ~D'T =T;Q (3) V. (3). *259-11.) F. Prop The two following propositions are lemmas for *273-22-212. *259-17. F:. WAA*=A:SE(Aw*A)'AAP(IW.:)s. Dem. U'iUWSA:)U(A A)Al1 F. *250-242. *257-35. *259-1. F:.Hlp.S,Te(Aw*A)'A&.StT.:): A'SC-T.v. Awl'TCS: [(*259-02)]): U' W'S C U1T. v. U' W'T C US: [Hp] D:U'S +U'T:.) F. Prop *259-171. F:. W AA* =A: S (A w*A)'IAAU'LIW.),s. D'Srt D'W'S =A: D. D r (A w*A)'1A 1I-+ 1 [Proof as in *259-17] *259-2. F: W = X?{X= seqp'D'T4I seqQ'U'T}) WA El -+),1. W AA*=A Dem. F.*206-2.:)F:.Hp.:):TeU ['W.:).D'TAD'W'T=A.UI"'TAU'WT=A (2) F. (2). *55-134.) F:Hp. T U(VW..(W'T CT) (3) F. (1).(2). (3). *259-15.:)F. Prop

106 SERIES [PART V *259-21. F:Hp*259.2.Q2iJ.:).W AQCP. DI WAC CIP.,(fWAC C`Q Dem. F.-*206'133.) F: Hp T eG'1W.:).(WIT);Q =A (1) F. *206-21.) F: Hp (1). D. seqQ'GI T'-% e Q"U'~T. [*37A461]:).(WIT) IQ IT=A (2) F. (3). *41-43. *258-242.) F: Hp.D. D'IYACG'IP (4) Similarly F: Hp. D. iWA C C'Q (5) F. (4). *206-132. D I-: llp (). Te(Aw*A)'A.). seqp'D'Tfep'P"D'T. [*40-16].seqp'D'T ccT"seQ(T [*40-67] D. (T"QjseqQ'121'T) T~ t'seqp'D'T C P (6) F.(I). (2). (6).)F: Hp (1). Te(A w*A)'A. T;Q CP)(A w'T);Q CP (7) F.*259-111.)F: C(~A' c{~xQ.: (ST). Te X. x(T;Q)y:. [*11v62.*10-23] T: TeX.:)T. T;Q C P: D. xPy (8) F.(8). Comm.)F:.XC(Aw*A)'A:TeX.)T.T;QC-P::).(~'X);QC-P (9) F.(7). (9). *258-242.) F:. Hp.):TE(Aw*A)'A.&.). T;Q C-P: [*259-11]):WA;Q CP (10) F.(10).(4).-(5). DF. Prop *259 W; 211. F:Hp*259-2.P2C-J..WPC Q [Proof as in *259-21] *259-22. F:

http://quod.lib.umich.edu/cgi/t/text/text-idx?c...stmath;rgn=main;view=text;idno=AAT3201.0003.001 (56 of 246) [5/26/2008 7:31:50 PM]

HP *2~59-2 P econnex.:). D"(Aw~*A)'A CsetctP Dem. F.*211P22.:)F:Hp. Te(i'W. D'Tesect'P.). D'Aw'Tesect'P (1) F. *211P63. D F: D"'X C sectP.) D. D'gc'X e sectP (2) F. (1). (2). *2-8-242.:) F. Prop *259-221. F: Hp*259-2. Qeconnex.). U"l (Aw*A)'A Csect'Q *259-222. F:Hp*259-2.PeSer.E!B,'P.Q2'CJ.Te(Aw*A).A.:). T;Qe C'Ps [*259-2122. *213-161] *259-223. F:llp *259-2.-Q cSer. E! B'Q. P2 CJ. T e(A w*A)'A. *259-23. F:Hlp*259-2.P,QeSerriU'B.Te(A w*A)'A.:) (SM, N).Me'PT,. Ne C'Q. T~eMsmnior N [*259-2-21222-223].

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SECTION D] INDUCTIVELY DEFINED CORRELATIONS 107 *259-24. F:. Hp *2592. P, Q e 2.): D' W= C'P. v. WA = C'Q Dem. F. *206-18. F.: Hp.P=A.:.. WA=A (1) F. * 20618.:) F: Hp. Q =A.:). WA = A (2). (1).(2). F:.Hp:P=A.v.Q=A:):D 'WA= C'P.v. aWA- C'Q (3) F. *259-11. *257-36. F: Hp. [! P. Q! 'Q. D WA e D'AW. [*259-12]). (E! seqp'D' WA. E! seqQI' WA) (4). (4). *252-1. *259-22-221.) F:. Hp. [!P. g! Q.:): D' WA = C'P. v. (a'WA = C'Q (5) F. (3). (5). D F. Prop *259-25. F:. Hp 259-24.:): (,BI). / e sect'Q. WA e P smor (Q /). v. (ga). a e sect'P. WA e (P C a) smor Q [*259-23-24] The above affords a new proof of *254'37, which asserts that if P and Q are well-ordered series, one must be similar to a section of the other. In virtue of *259'25 (which has been proved without using the propositions of *254), WA is the correlator which correlates the whole of one series with part or the whole of the other. It will be observed that the relations (Ay*A)'A are the class of correlators of sections of P with sections of Q, provided P, Qe 12 - t'A; i.e. F: Hp *2592.P, Q e - tA.. (Aw*A)'A= T {(aM,N). Me C'P,. Ne C'Qs. Te Msm-or N}.

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SECTION E. FINITE AND INFINITE SERIES AND ORDINALS. Summary of Section E. In the present section we shall be concerned first with the distinction of finite and infinite as applied to series and ordinals. We shall then establish the distinguishing properties of finite ordinals, and shall deal with the smallest of infinite ordinals, namely o, the ordinal number of a progression. Finally we shall briefly consider certain special ordinals, and the series of cardinals applicable to well-ordered infinite series, namely the series of "Alephs," as they are called after Cantor's usage. In dealing with the finite and the infinite as applied to series, we have constant need of the relation (P,)po, where P is the generating relation of the series. We have x (PI)po y. - P (x i- y) e Cls induct - t'A, i.e. "x (P,)o y" holds when, and only when, there is a finite number of intermediaries between x and y. When P is finite, we have P = (P)po, but we may have this when P is not finite. The infinite series for which this holds are progressions and their converses (which we will call regressions), and series consisting of a regression followed by a progression, of which an instance is afforded by the negative and positive finite integers in order of magnitude.

*260. ON FINITE INTERVALS IN A SERIES. Summary of *260. In the present number we are concerned with the relation which holds between x and y when the interval P (x - y) is an inductive class other than A, or when the interval P (x H y) is an inductive class of at least two terms. This relation holds if x and y have any relation of the class fin'P (defined in *121). We will call this relation Pfn. Thus we put Pfn = 'fin'P Df. Then xPfny holds when xPy, where v is an inductive cardinal other than 0 (*260'1). This relation will take us from x to any later term which can be reached without passing to the limit. But if in the interval P(x-ry) there is any term which has no immediate predecessor, i.e. any member of C'P-G'P,, then we shall not have xPfny. Thus Pfn confines us to terms which are at a finite distance from our starting-point. We shall find that if Pe f, a necessary condition for the finitude of P is P = Pf. This is not a sufficient condition, since it does not exclude progressions, but these are the only infinite series it admits, and these are excluded by the assumption E! B'P. Although Pfn is not in general serial when P or Ppo is serial, it becomes serial when confined to the posterity or the ancestry or the family of any term with respect to itself (*260'32-4). When a series P is well-ordered, the whole series can be divided into constituent series, each of which is the family of any one of its members with respect to Pf. (except when P has a last term which has no immediate predecessor, in which case this last term must be omitted). (Cf. *264.) Each of these series (except the last, possibly) is a progression, and the last is either finite or a progression. Hence every infinite well-ordered series consists of a series of progressions followed by a finite tail (which may be null); hence the cardinal of the field of an infinite well-ordered series is a multiple of 0. These results will be proved later; for the present we are concerned with the proof that the family of any term with respect to Pfn is a series of which the generating relation is Pfn with its field confined to that family.

110 SERIES [PART V In the present number we are chiefly concerned with the relations of Pfn to P1. We have *260'27. F: Ppo e Ser. 3. Pfn = (P,)po This proposition will be used very frequently throughout this section. Without any hypothesis we have *26012. F Pfn PPo We have also *260'15. F. Pfn = (Po)fn Hence whatever properties of Pfn result from the hypothesis that P is a series will result from the weaker hypothesis that Ppo is a series. If Ppo is a series, Pfn is contained in diversity and is transitive (*260'202), but not in general connected. In comparing Pfn and (P1)po, we constantly need the proposition *260'22.: Ppo e Ser. D.(P),= P. P e -- 1. (Pl)po J From *260'3 to the end of the number, we are concerned with the result of limiting the field of Pfn to the ancestry, posterity or family of some member of its field. We have *260-33. F: Ppo eSer. x e D'P. P = R.). 44- 4- 4 -Pfn. (tle X P fn") = (R*'$) 1 Rpo = m(RX) 1 R}po = JR r (Rpoi')}po *260*34. F: Hp *260-33.. {Pfn ('x v Pfn'x)}I = (R,'x) 1 R = R R Rpo' *260-01. Pfn = s'fin'P Df *260'1. F: XPfn y.. (v). v e NC induct - '0. xPIy [*121-121. (*260-01)]

*260'11. F: xPfn y.. P (X M Hy) e Cls induct - 0 - 1 Dem.. *2601. *12111.: xPf y. =.(v). eNC induct- t'O. P(x y)ev+01. [*120-472] =. (g[). e NC induct - 'O- 'I1. P (x y) e. [*120'2] =-. P (x H y) e Cls induct - 0 - 1:) F. Prop *260-12. F - P, Ppo Dem.. *121-321. *117-511.): I e NC induct- '0.. P, G Pp (1) F.(1).*260-1.hF.Prop

SECTION E] SECTION E]ON FINITE INTERVALS IN A SERIES11 ill *260-13. I-: xPfy.).P(xH-,y), P(x —Hy)ECls induct-t/A Dem. F. *91-54. (*1 21-011-012-013).) -. P (x ~- y) CP (x i-y).P (x -i y) CP (x - y). [*120-481.*26011I])F: Hp.). P(xi — y), P~x —y)ECls induct (2) F. (1). (2).)DF. Prop *260-131. F:.PPOC-J.):xPffly.=-.P (xF —y)eCls induct-i/A. P (x — i y) E CIs induct - i/A Dem. [*120-201]:).P(x~4y) eCls induct (2) [*52-41]:).P(xi-.i Y) 6 0v 1 (3) F. (2). (3). *260-11)DF::Hp. Hp (1). D. xPffly (4) Similarly F: Hp. P (x — q y) e Cis induct.). xPf.y (5) F. (4). (5). *260,13.-) F. Prop Dem. F.*121-52.) F: Hp.). hfinid'P =P*. [*121~302] - P* I rU'CP [*91~541] =Pp0 D)F. Prop *260-15. F Pfn = (P50)fn [*260-1. *121~254] *260-16. F P~ =P [*260-1. *121~26] *260-17. F: PpoecSer. xPpoy.).P(x ~.-y)=C [PC~~y x =B'{P~.~P (x -.iy).y=B'Cnv'tPp. P (x i-iy)1 Dem. F.*121~242.:)F: Hp.) x, y eP(x i-iy). xt+y.(1 [*52-41] xi.I e [*202-55] OAC'P. VP (x ~~ y)J = P (x i-i y) (2) F.*91~542.)F:. Hp.):zeP(xi —Iy).z+x.).xtPp0~P(xi —y)}z: [(1).*205 35]) = Mill {PP0 ~ P (x ~-q Y)}'fP (x i- Y [(2).*205-12] x: = B'1Ppo ~ P (xi —i y)} 3 Simiilarly F: Hp.. y = B'Cnv'{JPP0 ~ P (x i y)} (4) F. (2). (3).-(4). D F. Prop

112 SERIES [PART V The following propositions are concerned in proving that if Ppo e Ser, Pfn = (P1)po and P, = (P,),. Note that ' x (P1)poy " means that we can get from x to y by a finite number of steps from one term to the next, so that the series contains no limit-points between x and y. The relation "x (P,), y" means that v -c 1 intermediate terms Z1, Z2, Z3,... Zv-ol can be found, each of which has the relation P1 to its neighbour, and such that xP1z and z,_,,Ply. Thus we have to prove that, provided Ppo is a series, this occurs when, and only when, the number of terms in the interval P(xyi-y) is + 1. *260'2.: Ppo e connex. P. y. yP*z.. P (x H z) = P (x H y) v P (y H z) Dem. FI-.201'1415.:): Hp..P(x Hy)CP(x H). P(y Hz)CP(xHz) (1) F. *202-13-103. F:. Hp. xP*w. D: wPy. v. yP*w (2) F. (2). *121-103.) F:. Hp. w E P (x - z). Pw: P.: wP*. v. yP*w. wPz: [*121'103]): w e P (z H y) v P (y Hz) (3) F. (1). (3).)- F. Prop *260'201.: Ppo e connex..Pfn e trans Dem. F. *260-12. F: xPfy. yPfz.). xP*y. yP*z (1) F. (1). 260-2.) F: Hp. Pfy. yPfnz.. P(x z) = P (y) v P(yHz). (2) [*260'11.*120'71]. P (x Hz) e Cls induct (3) F. *60-32'371.) F: ae O v 1. I C. C. eO 0v 1: [Transp] D -: e0 u 1. 3Ca.). ae O v 1 (4) F. (2). *260'11.) F: Hp. xPfny. yPfnz. D. P (x H y),e 0 1. P (x Hy) C P (x Hz). [(4)]). P (x-Mz)eO 0 v 1 (5). (3). (5). *260-11. F: Hp. xPfy. yPfz. D). xPfz:) F. Prop *260'202. F: Ppo e Ser. 3. Pf, e R'J n trans Dem. F. *,260-12. } F: Ppo C J. D. Pf, C J (1) h. (1). *260201.) F. Prop We shall not have in general Ppo e Ser.. Pf, e Ser, because Pfn is in general not connected. Pf, only relates two terms

which are at a finite distance from each other, and hence divides Ppo into a number of mutually exclusive parts. We shall only have Pf, e Ser when every interval in the series is finite.

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SECTION E] SECTION E]ON FINITE INTERVALS IN A SERIES13 113 *260-21. F: P50 eSer.xP~y. yP~z.:).P(x~-4z)=P(xI.-4y)vffz Dem. F.*121P242.)F:Hp.:).yeP(xi-4-y) (2) F.*260'2.:)F: Hp.). P(xi-4z)=P(xI —iy) vP(,y~-.z) [(1).(2)] =-P(xi-qy) v fz:)F.Prop *260-22. F:P1,O USer. D.(PI), =PI.I I (IpoCJ Dem. F. (1). *204-7.)F: Hp.). PE1I-+ 1 (2) F.*121P305.:)F:Hp.).PjC-P. [*91-59]D.(jP PO F. (1). (2). (3). *121-31.) F. Prop *260-23. F:Ppo ESer. vcNCinduct.:).(P,4,el —+I [*121P342. *260-22] *260-24. F:P0 e Ser. ve NC induct. x(P4,,y. x(P1)~,+. y, Dem. F. *121-35. *260-22.:) F: Hp. D. x {(P,), P,} z. [*260O23.Hp]:). yP z::) F. Prop *260-25. F:Pp0 ESer. R-PI xR*y.:).P(x~-4y)=R(xi —iy) Dem. F. *260-24.) F: Hp. veNC induct.xR ~ R+,zPx —y=~~qy.) &yRz. P (x ~~ y) =1?R (x F — y). [*260-21])P (x ~z) = R(x I-. 4Y) v fz [*26O-22.*121 371-304] = 1? (x -iz) (1) F. (1).)D F:. Hp. ve NC induct: xR~y.:)y P (xI)=1(x i-4qY):) R~,1z. P (xi-.iz) = R(x i —iz) (2 F. *121-301-22-242. DF:Hp. xRy -D.)P (x -4 y) = fx = R(x I —y) (3) F. (2). (3). Induct. D F:. Hp.): v e NC induct. xR,,y) D. P (x F- y) =B (x F-i~y) [*121P52. *260-22] D: xR*y. D. P (x -4y) = R(x i-iy):. D F. Prop In the above proposition, " Induct " refers to *120-13. The"" of *120-13 is replaced by R. & W. III. 8 x -- y = R (x F -i) R. & W. Ill. 8

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114 SERIES [PART V Thus (2), in the above proof, is (when v is replaced by ~ and (3) is 400. Hence, by *120,13, we have ~e NC induct. Oa i.e. v E NC induct.-::X' xly.)0 R (x i-.i y), which is the inference drawn in the above proof. Wherever " Induct " is given as a reference, it indicates a process such as the above, making use of *120-13 or *120-11. *260-251. F:PPO e Ser.)(P,)po C Pf. Dem. [*121-40.*26O-22]).P (x ~-q y) E Cis induct (2) F. *121,242. (1). *260-22.) DF- Hp (1).) x, yeP (x -q y).xt+y. F. (2). (3).) F: Hp. w (P1)poy. P (x -i-~y) e Cls induct - 0 - 1. [*260-11] X.f PY::) F. Prop *260-26. F:.P0 eSer.R=P,.xi?*y.):xP1,y. u.wR1,y Dern. F.*260-25.)F:. Hp.): P (xi —4y) =R B-y [*121-11]) xvP~y. -=. xl~y:.) F. Prop *260-261. F: P0 E Ser. v E NC induct - tf0. xP~y. xP,+,1.).y~ Dem. F.*121-11.)F:Hp.).Nc'P(xn-ig)=v+,,1.Nc'P(x~-ez)=v-,-02. (1) [*I120'32].y + z (2) F. (1). *1 20-428. F Hp.). Nc'P (x ~i z) > Nc'1P (x i —i y). [*117-222. Transp] - (Z ~ j [*202-103]). yPoz. [*202-171] -P(-)=P(-)VP(-) [*1 20-41.(). (3)]. P (y -'q z) E 1. [*121-11] D.yP'z:) F. Prop *260-27. F PPO0 E Ser. Pf. =Ps,, P Dem. F. *260-261.)F:Hp. v' NC induct - tf0. xP1,Y. xPv+e,,Z. x (P1)P0.).

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SECTION E] SECTION E]ON F1INITE INTERVALS IN A SERIES11 115 1-.():(). *1p4. v) NC Hp idc t':xvEPCinucy. t0)y~(Pp,,y C P1: [*260-1]): Pfn C- (P1)p0 (4) F. (4). *260-251 F.)I-Prop *260-28. F-:Pp E Ser. v eNC induct -t"'.. P = (P1),= (Pfn),, Dem. F-.*260-1. D -Hp. xP3,y. D.-XPfnY.[*260-27]:). w (P1)~,.y (2) F-.*1 21254.:)F-. PV=f(O1 [*260-27] F F: Hp.). (PI)v = (Pf0,), (5 F-. (4). (5).) F-. Prop The above proposition does not hold ini general when v =0, for if P is a compact series, P, = A, so that (PI)0 A but P0O I r G'P. *260-29. F-:P50 ESer.xPf,, y.).P (xn —y)=P,(x~-4y)=Pf0,(xI-y) Dern. F-.*960-2725.)F-: Hp.).P(x~.-y)=:PI (x~-y) [*I 12253.*260-27] =Pf, (x i —y):) F-. Prop The following propositions are mainly concerned with the result of confining the field of Pf. to the posterity of a single term. *260-3. F-: P0 eSer.:).D'Pf -=D'P1. U'Pf0 =IPP1.CGPf0 =C'P1 [*260-27. *91-504] *260-31. F-: P10 eSer. x eD'P.) C'{ fPfn1 (t/ CXv Pfj'X)} (PI) *Ix = Ixv Pf0'Ix Dern. F- 6 *2602 7D F-: H p.). t'fx vPf X = t/ "Xv (OO [*96-14] = (P1)*'x (1) F-.*260-3.:)F-:Hp.).a!Pfn'X4 -F-.*36-13.) F-:yEPfn'X.:).X{Pfn~ (t'XVPfn'X)}Y. 4- +~~~F.()(3).) F-: Hp.)D. t' X W PM 'X C C'Pfn, (t'X VPfn'X)}. +- 4 -[*37-41] D. L'IX V PfnI'X =G'Pfn ~ (t'X V Pfn'w)} (4) F-. (1). (4)..) F-. Prop 8-2

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116 SERIES [PART V *260-32. F: Ppo eSer.).D Pfn (tIX v Pfn'X) = Ppo ~ (t'X v Pf.'X). Pfn ~ (t'X V Pfn'X) E Scr Dern. F-. *2 6 0 12.)F- Pf, (tfx V Phf0'X) C- Pp. ~ (Xv Pfn 'X)(1 F-.*260-3. *200-35.) F THpT x-T, D4-Pn~W Pn() Po~WXvP.x 2 F. *201V521. *260-27.) F: Hp. x E D'P1.) Pf1 ~ (t' x VPf0'x) =(P1)1 (P1)*(x. [*202-14.*260-22]:). Pfn ~(t 'X v Pfn1'x) E connex. [*260-202]) Pfn ~(t'Cx v Pfn'x) E Ser. (3) [(I).*260-31.*204-41]:). Pf" ~(t'x u f')= + ('xvP0 (4) F-. (2). (3). (4).:)F-. Prop *260-33. F-: Pp e Ser. x eDIP,.P, =R. D. Pf11 ~(L' X Pfn'X) =(R*'"x)I Rpc, = {(R*'x)I R}p0 = JR? r~(RPO'X)1PO Dem. F-. *260-27431 F-: Hp. Pn ~(t'X V Pfn'X) = Rpo ~ R*'x [*96K16.*91P602] = (R*'fx)1 Rpo (1) [*96-2.*260-22] = JR r(RTh I')AO (~3) F-(1).(2). (3).) F-.Prop *260-34. F-:Hp *260-33.:). {Pfn ~(tf x vPfn'x)} = (1R*'x)1 R=1 Rr~p'x Dem. F-.*260-33.-*121P254.) F-:Hp.:). Pfn ~(fX V Pfn'X)J1 = (R*'x)1 R}1 = {R Rp'x}, (1) F-. (1). *121P31. *260-22.:) F-. Prop The following propositions are concerned with the result of confining the field of Pf,, to a single family. *260A4. F-: PP0 E Ser. Pf0, ~Ppf'xe Ser. C'(Pfn ~ Pfn'X) =PfnX= (Pi)*'X.- Pfn'X ' E Dem. F-. *260-27.*97-17)F-: Hp)Pfn Pfn'X =(PI)po ~ (PI)*'X.[*202-15.*260-22] Pfn P fn i 10x e connex. [*260-202.*204-42].Pfn ~Pfn IX eSer()

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SECTION E] SECTION E]ON FINITE INTERVALS IN A SERIES11 117 F. *9 7 18. -) F. d'((Pf1 ~ Pffl'x) = Pf0,'x (2) F.(2). *260-202.*200-12.)FHp)Pf0'X "e 1 (3) F *260-27. *97-17. F: Hp.) P0X=()*x(4) F.(1).(2). (3). (4). DF.Prop *260-41. F: Pp0 E Ser. R-P,:)D Dem. ~~~Pfn ~PfnX =Rpo, RI*'CX= (R*'X) 1 RP Rp0 rR*'x F*260-27. *97-17.)F:Hp.)Pfnf Pfn'X = Rpo ~R*'X (1) F.*97-13.)F:Hp.,yeft*'x.yR z.:). zeR C"R'xvR '"R*'X. [*92-31 1.*260o22]).z C R*'x v R*'x. [*97-13.*36-13]

(RP0 ~R*'1x) z (2) F.*35-21441.:) F. R0 R'C(Rw)R0 (3) Similarly F: Hp.. RPO R ft'x = ftO 0 LI*'x (5) F.(1).(4).(5).) F. Prop *260-42. F:Hp *260-41.).Pf ~ Pfn'X (R*'xl R)p. = (ft r R~'x)p Dem. F.*92-32.*260-22.)F:Hp.).R""R*'xCR*'x. *4 *4+ Similarly F:Hp. Rp- B1 tR*1x = {R r R*'x1 p (2) F. (1). (2). *260-41..Prop 1I260'43. F:Ppe eSer.) Dem. t~~~~ffnr' f P0X~I PI = i Pf11 x = (Pf0 'x) 1P1 PI r (Pf0'x) F. *260-42. *121 —254.) [*121P31.*260-22] = (1?~'x) 1 ft [*97-17.*260-27] =(Pfn'Cx)1 P1 (1) Similarly F: Hp. - { Pfn r Pfn"X~I = P1 t Pffl'X (2) F - (1) - (2). *35-11. -) F: Hp.:). {Pfn r Pffl'XI =P1 rPfn'X (3) F. (1). (2). (3).:) F. Prop +*4* Observe that the two series Pfn rPfn'x and Pfn rPfn'y are either identical or have no common terms in their fields. This results immediately fromi *97-16, since the, fields of the two series are (P1)*'x and (P1)*',y.

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*261. FINITE AND INFINITE SERIES. Summary of *261. In this number we define finite and infinite series, and we show that, where well-ordered series are concerned, there is only one kind of finitude, i.e. there is not the distinction, which exists in cardinals, between "inductive" and "non-reflexive." We also give various equivalent forms of the distinction between finite and infinite series, and some of the simpler properties of each. The propositions of this number are numerous and important. We define an infinite series as one whose field is a reflexive class, and a finite series as one which is not infinite. Thus we put Ser infin = Ser n C"Cls refl Df, /f infin = 1 n C"Cls refl Df, Ser fin = Ser - Ser infin Df. 2 fin = D - l infin Df. We also put, to begin with, 12 induct = nA C"Cls induct Df, but in the course of this number we prove *261-42. F. D fin = f induct so that the symbol "1 induct" is not required after the present number. After some preliminary propositions, we proceed (*261*2 ff.) to various criteria of finitude and infinity. We have *261-25. F:. PeSer.): C'P e Cls induct - 'A.. P = Pf. E! B'P. E! B'P The condition P = Pfn insures that every interval is finite, but this still leaves it possible for our series to be a progression, or its converse, or the converse of a progression followed by a progression (i.e. the type of the negative and positive finite integers in order of magnitude). The third of these

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SECTION E] FINITE AND INFINITE SERIES 119 possibilities is excluded by either E! B'P or E! B'P; the second is excluded by E! B'P, and the first by E! B'P. We have *261-212.:. P e f.: (I 'P = (P -. P = (P),. -. P = Pfn " C('P1 = (IP " means that every term except the first has an immediate predecessor. We have *261'26. F: P e Ser. a C C'P. g! a. a e Cls induct. D. E! minp'a.E! maxp'a and *261-27.:. P eSer: a C'P.! a.). E! minp'a. E! maxp'a:). P = Pfn. C'P e Cls induct whence we obtain *261-28. F::PeSer.:. a C C'P. a! a. Da. E! minpa. E! maxp'a: =. 'P Cls induct I.e. a series whose field is inductive is one in which every existent subclass of the field has both a minimum and a maximum. From the above, together with an inductive proof that every inductive class which is not a unit class is the field of some series, we obtain *261-29. F. Cls induct= 1 v CG"P P e Ser: a C ('P. X! a.). E! minp'a. E! maxp'a} =1 v C"((I n Cnv"I) The above proposition is interesting as giving an alternative method of treating inductive classes. Instead of the definitions adopted in *120, we might have taken the above proposition as the definition of inductive classes, putting NC induct = Nc"Cls induct Df. We should thus wholly avoid the use of mathematical induction in definitions; hence if such avoidance were in any way desirable, it could be secured by dealing with series before introducing the distinction of finite and infinite, and then defining inductive classes as the fields of series which are well-ordered backwards as well as forwards. The inductive properties of such classes would then be deduced from *261*27, together with *260'27, in virtue of which we have P e f n Cnv"n.. P = (PP)po, whence, by *91-62,:: P e v n CnvpQ2.:. xPy. -: P"h C i. Po x e lo. De. y e i. In virtue of this proposition, if ry is the field of a well-ordered series P whose converse is well-ordered, then any property which is inherited with respect to P, belongs to all the successors of x (where x e 7) if it belongs to the immediate successor of x. Hence mathematical induction follows.

 From the above we obtain at once *261-31. F:. P e Ser. Z): C'P e Cls induct. -. P, P e f I.e. series whose fields are inductive are the same as inductive wellordered series, and are also the same as well-ordered series whose converses are well-ordered. Hence also we obtain *261-33. F: P,Qf. Q cP.). Qe induct I.e. a descending well-ordered series of terms chosen out of a well-ordered series must be finite. This proposition, which is due to Cantor, has been used by him in many proofs. We have *261'35. h:. P e. D): C'P e Cls induct - t'A. -. (aPP = (I'P. E! B'P In *253*51 and following propositions we have already had the hypothesis ('PP = ('P. E! B'P, which now turns out to be equivalent to the hypothesis that our series is finite and not null. Thus we have *261 36. F:. P e t. D: C'P e Cls induct - A. -. Nr'P 4 i - Nr'P *261'4 and following propositions are concerned in proving that a wellordered series which is not inductive always contains progressions, and in deducing consequences from this. We have *261-4. F: P e - f induct. (P)*'B'P} 1 P1 e Prog The above proposition is very important, for many reasons. One of its most important consequences is that, if P is a well-ordered series which is not inductive, its field contains an K,, and is therefore a reflexive class (*261'401). Hence a class which can be well-ordered is either inductive or reflexive (*261*43), and a well-ordered series is either inductive or infinite according to the definitions given above (*261'41). Hence where wellordered series are concerned, the two ways of defining finite and infinite (namely those in *120 and *124) give equivalent results. This cannot (so far as is known) be proved for classes in general without assuming the multiplicative axiom. From the above-mentioned propositions it follows that a well-ordered series is one in which P1 limited to the posterity of B'P with respect to P, is a progression in the sense of *122 (*261'44), and that any class contained in a well-ordered series is either inductive or reflexive (*261*46). The number ends with some propositions in ordinal arithmetic. We prove that PQ is well-ordered if P is well-ordered and Q is a finite well-ordered series (*261'62); that if R is a finite well-ordered series, and P is less than Q (in the sense of *254), then pR is less than QR; and that a finite well-ordered series is less than an infinite one (*261'65).

SECTION E] FINITE AND INFINITE SERIES 121 *261 01. Ser infin = Ser ^ CG"Cls refi Df *261 02. Ii2 infin = n C"Cls refi Df *261 03. Ser fin = Ser - Ser infin Df *261-04. &2 fin = f2 - f2 infin Df *261-05. fl induct = fn C"Cls induct Df *261-1. F P e Ser infin.. P E Ser. C'P c Cis refi [(*261'01)] *261-11. F: Pei2infin. =. P ef2. C'Pe Cls refi [(*261P02)] *261-12. F:PeSerfin. =n.PeSer-Serinfin..=PeSer.G'Pc. eClsrefl [(*261P03)] *261-13. F:Pe fin.-n.Pef -f2infin.n-.PEfl.G,'Pc e Clsrefl [(*261 04)] *26-14. F P E fl induct. -.P e f2. O'P e Cis induct [(*261P05)] *261-15. F: PeSerinfin.PsmorQ.). QeSerinfin Dem. F. *261. F liHp.). P e Ser. G'P e CIs ref. P smor Q. [*204-21.*151-18] Q. Q eSer. C'P e Cls refl. C'P sm '6Q. [*12418]). Q E Ser. C'Q e Cls refl. [*2 61 1] Q.QeSer infin: F. Prop *261-151. F P e Ser infin.). Nr'P C Ser infin [*261-15] *261-152. F: P e Ser infin.. Nr'P C Ser infin. _! Nr'P A Ser infin [*261151. *155-12] *261-153. F: PE Ser infin. r. (gQ). P smor Q. Q e Ser infin [*26115. *151-13] *261-16. F: P ef2 infin. P smor Q. D. Q e f2 infin [Proof as in *261-15, using *26111. *251-111. *151-18 *124-18] *261161. F:P efinfin..Nr'P C finfin [*261-16] *261-162. F:Pe 2inhin. -.NNr'PCI infin.=-.a! Nor'PASerinfin [*261-161. *155-12] *261-163. F P e f2 infin=. (1Q).P smor Q. QE fl infin [*261-16. *151-13] *26/117. F: P e Ser fin. P smor Q..)Qe Ser fin [*261'15. Transp] *261'171. F: P e Ser fin. Nr. NrP C Ser fin [*261-17] *261-172. F: Pe Ser fin..N,,rP CSer fin..!N,,r'P Serfin [*261-171. *155-12] *261-173. F: PeSerfin.E. (aQ).PsmorQ. QESerfin [*261-17.*151-13]

122 SERIES [PARr v *261-18. 1-:Pe6f~fin.PsmorQ.:). Qef~fin [*261-16.Transp] *261-181. F: P e fi1 fin.:). Nr'lP C f2 fin [*261-18] *261-182. F: P e fffn. =. Nr'P Cfl fin. =3'g!N~r'P ^ f2fin [*261-181. *155-12] *261'183. I- P effin.=. (RQ).P smor Q. Q e tfin [*261-18.*1'1-13] *261-19. F:Pe f2induct. P snor Q.D. Q e iinduct [Proof as in *261'16, using *120-214 instead of *124-18] *261-191. F:P e Q indu~ct.). Nr'P C Qinduct [*261-19] *261-192. F-: P E Linduct.. Nor'P C Liinduct.=. 2! Nor'P A Liinduct [*261-191. *155-12] *261-193. I-:P E Linduct.=. (, Q).P smor Q. Q eLiinduict [*261-19. *151-13] *261-2. I- P50 E connex. (B'P) Pf,, (B'P). D. CI'P e Cis induct Dem. [*2601 l.Hp].C'PE Cls ji ilu ct: F. Prop *261-21. F: P econnex.P=-Pf. E!B'P.E!BP.)C'PE Cls indUct Dem. F.*202-103. *93-101.)D F: Hp.). (B'P) P(B'P). [Hp]) (B'P) Pfn (B'P). [*261P2]).C'P e Cis induct: D F. Prop *261-211. F:P eSer.).minp'{[P'x -(P1)Pjx}C P'P - (UPI Dem. F. *91P51 1. *1 21'-305. D F:. Hp.): yEP'x A(P1)50'x.yPlz.). ZE6P'te A),I 4- 4 — 4- 4 — [Transp]): z E Px - (P1)5ox._yPIz.:). y e - P'x u - (P1)50'x (1) F. *91-502. D F:. z eP'x -(Pl) p0'x.): z eP' - Pl'x: [*201-63]):Hp.3.xP~z (2) F.*201-63. F: Hp. xPlz. YPIz.) r (YPx). y f X. [*202-103]).xPY (3) F. (2). (3).)F:Hp.z E P'x - (P1)50'x..yPJz) y E

P'lx PA Y,6 PIX ~- 4 —Pl~ [*2O1-63]).y E PIZ A {P'x - (P1),,ia4. [*205-14]:). Z % emninp'{P'wx- (P1)p0'x} (4) F.(4). Tra-nsp. F: Hp. z eminp'tIP'x - (Pj)50'x}.j (d [y). yPj z: F. Prop

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SECTION E] SECTION E] FINITE AND INFINITE SERIES12 123 *261-212. F:. P ef~.):cIP, = (PIP P =(P,),0., P =Pffl Dern. F.- *121L305.) F: Hp.:). (P,),, C- P(1 [*32-18]).(ax). a! P'x - (Pl)Wjx. [*250-121]:). (ax). E! minp'jP'x - (P1)P,0'x} [*261'211] D. a'I'CP - ii'p1 (2) F.(2). Transp. F: Hp. (PP=(TPP,. D P =(P,)p, (3) F *91'504. F::P =(P1)PO.:).P(P = wPP (4) [*260-27].P= Pf,,:. DF.Prop *261-22. F:P eSer. C'P eCl,-in md ct.).P =Pfn. D'P D'P1. IPP IPP1 Dem. F. *260K12. *20118. D F: Hp. D.Pf,,C P (1) F.*121V242. D F: Hp.xPY. D.x, YeP (x i-4 y). x Y. [*52-41] D. P(x ~-4Y).E 0 1 (2) F.*1 20-481. DF: Hp.:). P(x i- y) eCis induct (3) F.(1).(4).) F: Hp.D. P= Pffl. (5) [*260-3])D'P DIP,. PP=U'PI (6) F.(5)(6). D)F. Prop *261-23. F: PeSer.D'P,=D'P.r-..E!B'P.fj!P.).C'PeClsrefl Dem. F. *91-52.:) F. Pi cc(PI)*'Cx =(P1) P0x (1) F.*91P54. *260 22.) F.*93-11. DF:Hp.:).D'P,=C'P. (3) [*90.18]. (P1)*'x CD'P, (4) F. *260-22.) F: Hp.). Pl ell —) 1 (5) F. (1). (2). (4). (5). *73'21. *91 74.)D F:. Hp. D: x e Ci'P. a). (P1)*'X sm (P1)50'x. (P1),0'x C (Pi)*'x. 4 — 4 — a! (Pi)*'x - P).X [*1 24'1 6]).(P,)*'x e Cis refi (6) F. (6). (3). (4).:) F: Hp.:). a! Cis refi rn CIC'P. [*124-141] D. C'P eCls refi:)D F. Prop

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124(SERIES [PART V. *261-24. F:PeSer.C'PeCls induct-t'A.). E!B'P.E!B'P Dem. F. *261P22.)F:Hp.).D'P D'PP. [*261'23.Transp] *).E!B'P (1) F.(l)P.)F:Hp.).E!B'P (2) F. (1). (2).:)Fl. Prop *261-25. F:.PeSei.): C'P eClsinduct-tA..P=Pfn.E!B'P. E!B'P [*261 22-24-21] When P=Pfn.E!B'P.,E!B'P, Pis a progression; when P= Pfn. E! B'P. E! B'P, P is a regression (i.e. the converse of a progression); and when P= Pf.. E E! BP.)E! B'P, P is the sum of a regression and a progression. These propositions will be proved in the next number. *261-26. F:PeSer.aCC'P.a[!a. aeCls induct.:).E!minp'a.E! maxp'a Dem. F.*20517.F: Hp.ael..I E!minp'a. E!maxp'a (1) F. *20255. F F: Hp. ao El.. a = G'(P a). [*261'24]). E! B'(P C a). E! B'Cnv'(P r a). [*205-42]. E! minp'a. E! maxp'a (2) F. (1). (2).) F. Prop *261-27. F:.PeSer: aCCG'P.g!a.:).) E!minp'a.E!maxp'c(). P = Pf. (Y'P e Cis induct Dem. F.*250121. F: llp.:).P6f. [*250'21] D. D'P= DP1. [*260-3].D'P = D'Pfn (1) F.(1).)F Hp.XPfny. yeD'P.).yeD'Pfl.-XPfnly. 4 -[*260-2O1]). yCe P""Pfn'x". 4-~ [*260,12.*20118]. y E P""Pf x(. 4f[*205'111]).y naxp'PfnL'X (2) 4 -F. (1).(2). Transp.:)F: Hp. e D'P.. naxp'Pf#,x B'P (3) F. *250-121-13. DF: Hp. ft! P.)D. E! B'P. C~~~s~~~l ~ ~ ~ ~. (B'P) Pfn (B 4(P)..

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SECTION E] FINITE AND INFINITE SERIES 125 [*260-11]).P (B'P H B'P) E Cls induct. [*202-181] D. G'P e Cis induct (4) F. *120'212. D F: P = D. C.PECls induct (5) F.(4).(5).)I-:llp.).G'PeClsinduct. (6) [*261P22].P= Pf (7) F. (6). (7).) F.Prop *261-28. F::Pe Ser.):. a C G'P. aj! a. E! minp'a. E! maxp'a: C. GP E Cls induct [*261'26-27] *261-281.: ry e Cls induct - 1.). ry e C"Ser Dem. F. *20424.)F. AEC"Ser (1) F.*52-22. D)F. A v L'x e 1 (2) F. *52'22. D F: x =,Y. D. t'x U ffy E 1 (3) F.*204-25. D F: xty.. Ct'wvy xw E Cl"Ser (4) F. (3). (4). D F. tlx w illy e v I' C"Ser. [*521] D F: F ey.).y ff7/y el v C"Ser (5) F a.*51 2. DF: y E CO"Ser. yEry.). Dy v L'ye C"ISer (6) F. *20451l. *161-14.)F:Prye C"'Ser.aR! y. y- ey). D '. y u te Ce"Ser (7) F. (6). (7).:) F: ye C"Ser. 5! y.:).vy v t'y e C"Ser (8) F. (2). (5). (8). D F: C" I v C"Ser. D.yv'yvelyIv C"Ser (9) F. (1). (9). *120-26. D F: y Cls induct. D. r e 1 v C"Ser: D F. Prop *261-29. F. Cls induct = 1 v CG"P {P E Ser: a C G'P. ~! a. D. E! minp'a. E! maxp'la =1 v CG"(f Cnvllfl) Dem. F. *261-281.D)F:.eyECls induct - 1.)D:(HP): Pe Ser. y = C'P: [*261P28]):(aP): P E Ser: a C C'P.! a.,,. E! minp'a. E! maxp'a: ry G'P: A [*376]:):rYeC"PtPeSer:aCC'P.a!a.).E!ming'a.E!maxp'aI (1) F. *26128.) F:. P ESer: a C G'P.-! a.),. E! minp'a. E! niaxp'a: ry = C'P:).rye CIs induct:. [*376]) F: rY E G"P (PE Ser: a C GP. a.). E! minp'a. E! maxp'a).D. e Cls induct (2) F. *120-213. DF. I C Cls induct (3) F. Cls induct = C' P {P E Ser: a C C'P.! a.).E! minpla. E! maxp'a} [*250-121] = G1"(fl A Cnv"f1). D F. Prop

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126 SERIES [PART V The following four propositions are immediate consequences of the propositions already proved. *261-3. F::PeSer.:. C'P e Cls induct.-: P e: a C CC'P. a. E! maxp'a [*26128. *250-121] *26131. F:. P e Ser. D: C'P e Cls induct.. P, P e Q [*261-3. *250-121] *261'32.. Ser n G"Cls induct = n induct = Q n Cnv"f2 [*261-3114] On account of this proposition, we do not introduce the notation "Ser induct " for "Ser n C"Cls induct," because a series whose field is inductive is a well-ordered series, and therefore the notation " 12 induct "gives all that is wanted. *261-33.: P, Qe 2.Q P. Q.. Qe2 induct Dem. F. *204'2. DF h: Hp. D. Q eSer. Q P. [*250'14] D. Q e Ser n Bord. [*2.50-12] D. Q e f. [*261-32] D. Q e 2 induct: D F. Prop This proposition (which is due to Cantor) is of great importance in the theory of well-ordered series. It shows that, however great a well-ordered series may be, any descending well-ordered series contained in it must be finite. (A descending series in a given series is a series contained in the converse of the given series.) *261-34. F: P e 1f. ('P = I'P. E! B'P.. C'P e Cls induct Dem. F. *250'23. *214-12. D F:. Hp. a C 'P.: E! iaxp'a. v. E! seqpa (1). *206'181. ' F: Hp. a C C'P. a! a. E! seqp'ac.. seqp'a e (PlI. [*204-7] D. E! Pl'seqp'a. [*206-451] D. E! maxp'a (2) F. (1). (2). D F:. Hp. D: a C C'P. [! a. Da. E! maxp'a: [*261'3]): C'P e Cls induct:. D F. Prop *261'35. F:. P e 2. D: C'P e Cls induct - t'A. -. ('P, = ('P. E! B' [*261-22-24-34] Observe that "(I'P = G'P. E! B'P" occurs as hypothesis in *253-51 and some succeeding propositions. Thus this hypothesis is equivalent to the hypothesis that the field of P is an inductive existent class. It follows that

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SECTION E] FINITE AND INFINITE SERIES 127 if P is an inductive well-ordered series, Nr'Pv = Nr'P, whereas if P is a well-ordered series which is not inductive, Nr'P, = Nr'P 1 i; also that *261-36. F:. Pe ~2.): C'PE Cls induct - tA..Nr'P t i-Nr'P [*253-573. *261P35] *261-37. -:. Psfl.: C'Pe Cls ind uct.i.j- Nr'P = Nr'Pj [*253-574. *261L35. *161-2'201] *261-38. F:. P e.(: CP e Cls induct - VA..Nr'P, = Nr'P: C'P h, e Cls induct - 'A.. Nr'P - Nr'P-j-i [*253'56. *2613-5] 4 --- *261A4. F: P e 1` - fl induct. D. t(P,)*'B'PI 1 P1 e Prog Dem. F.*~204-77.: IF:Hp. =P=P,.) RRe1 —i I(F. *120-212.) F:. Hp.: ft! P: [*250-13] D: E! BIT: [*250'21] D: R = P1.:. B,'Pe D'R (2) F.*260122.)F:Hp.R=P,1..1)R,0 'J (3) F.- *93-103. *202-52.) F: P e fl. R = PI, 2[! R*IB'P - DIP. D. BIP e -R*IBIP. [*93-101.*91P54]. (BP) Rp(B'P). [*260-27] (BP) Pfn (BP). [*261-2] C. O'PE Cis induct (4) 4 -F. (4). Transp. F F: HP. R = P1.).?*'fB'P C D'P. [*250-21] D. B*'B'P C D'R (5) F. (1). (2). (3). (5).:) F:Hp. R = P1.) 4 -R eI — +1. B'P D'R. r {(B'P) Rp. (B'P)}. R*'B'P C D'R. 4 -[*122-52] D. (R*'B'P) 1R1 Prog:) F. Prop *261-401. F:Pd2 -f2induct.).a! oA Cl'G'P.C'Ps ClsreH Dem. F. *2614. *1231. D F: Hp.).D'f(P,)*'B'P} 1 P1 e K (1) F. *121305. F F: Hp.). D'{(Pj)*jB'P} 1 Pl C 'P (2) F. 1) (). F Hp.:).! N, n CI'CCP. (3) [*124-15]).C'P E Cis refi (4) F. (3). (4).) F. Prop

128 SERIES [PART V *261-41. F-. -(1induct = A7infin [*261-401.*261 11.*124-271] *261-42. F. 111 fin = fl induct [*261-41. Transp. *124-271] We shall henceforth use " fl fin " in preference to "&il induct." *261-43. F. C'Plf C Cis induct v Cis refi [*261-401-14] *261-431. F:P e f2-fA) {(P1)*1B'1P} 1 P1 = Pi r Pfn1'B'P = PI ~ (tfB'P vPfnl'B'P) = (L'IB'P V Pfn'B'P) 1 P1 Dem. F.*250-1321.)FHp.).BP e D'P1. (1) [*260-31] t. fB'1P v Pf'B'P = (P1)*'B'P (2) F. (1). *260'27.)F: HP.) PM 'B'P= (P1)P0'B(P. [*260-34] P. r1 Pfn"~B'P = t(P%)*'fB'P} 1 P1 (3) [(2)] = (t',B'P V Pfn'B'P) 1P1 (4) F. (3).(4). *35-11. F: Hp. j.(PI)*'1B'PJ 1PI =PI ~(t'~PVPnB'P) 4 -F.(3). (4). (5).) F Prop 4 -*261'44. F:.Pe&.):P1rPfn'B'.PEProg.=. P~flinfin Dem. F. *123-1.)F: P efl.PjrPf1'B'PeProg.).:! N, Cl'C'P. [*124-15]). P 'e Cls refi. [*261-1].Pe f2infin (1) F. *261P4431-41.:) F: P e f infin. PI rP1Pfn'B'Pe6Prog (2) F. (1). (2).:) F. Prop *261-45. F. &2 infi n = 12 A- P P T f''P e Prog}1 [*261-44] *261-46. F:P e 1.). CIICYP C CIsinduct v Cisrefl Dem. F. *250-141. *202'55. F: Hp. aC CP. a 1e.). P ~aE 2. a=C'(P ~a). [*261-43]:). a e Cis induct v Cis refi (1) F. *120-213.:)F: ad e.a eCls induct (2) F. (1). (2). DF. Prop *261-47. F:.Pe12. aCC'IP.:):aeClsinduct.=-.a~eClsrefl [*261-46. *124-271] *261-6. F:. Pe I. C1P C 2. Nc'C'P = v.-)P. VJ'P e12: veNc induct - t'0O - t'1: D: Q e 12. C'Q C 2..Nc'C'Q = v:)Q. fl'IQ Dem. F. *204-272.:) F: Nc'1D'Q =1.Q e Ser.).Q e 2r.[*5"6-112] C.'Qe 2(1

SECTION E] SECTION E] FINITE AND INFINITE SERIES12 129 F-. (1). Transp.) F: Q e 2. G'Q C f2. Nc'C'Q= v +0 1.VENC induct-t'0-t'1l.:).D,'Q,-,EI (2) F. - 261'24.) F: Hp (2). D. E! BlQ. [(2).*204-461] ~.Q = Q ~ D'IQ -I. B' Q. [*172'32] H.J1Q sinor F1'I(Q ~ D'Q) x B'Q (3) F* *1 10-63.)F: Hp (2).).Nc'D'Q +I = v + 1. [*120-311])Nc'D'Q v (4) [(3).*251P55] D. H'IQ 6 12(5) F. (5). Exp.)D F:. Hp.): Qef~. C'QC fl1. Nc'G'Q= v~ 1. D). IIQf:)F. Prop *261-61. F:Pef~fin.G'PCfl.:). H'IPef2 Dem. F.*261[6.:)F::cfw.=n:Pefi1.CGPCfL.Nc'G'P=ui.)p.H'Pef2:.):. V eNc induct - '0 -t'1l.):4w.-).k0(v +0l1) (1) F. *200-12. F r(HP). Pe 2. C'P C fl Nc'G'P =1I. [*10-53])F:Hp (1).). 01 (2) [*550'54.*204513]): P eSer.GIP = t'Y vt'Z.D.[MY'Pcf2 (4) F.(2).(3).(5). F.p1.:0vt'v'.b..(+1 (6) [*120-13]:): a 6 NC induct.:). Oa ~ (7) F. (7). *13'191.)F: P e f. C'P C~ f. NcC''P eNC induct.)p.HI'PE f: [*261'14-42])F: P e fin. C'PC ii2.)p.fHIP e&2:)DF. Prop *261-62. F: P ef2 Qegf2fin.).PQ e f Dem. F. *251P51. D F: Hp. f! P.D. P4I;Qe f (1) F. *165-26. D F: Hp.:).GCP J;Q C f (2) F - (1). (2).(3). *261V61)F: Hp. ft! P. RIllP JQ 6 Ii. [*176-181-182] D. PQ%&2 (4) F. *176-151. *250-4.)F: P A.).PQEf2 (5) F.(4). (5).)DF.Prop R. & W. III.9

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130 SERIES [PAIRT V *261-63. I-:E!B'R.PC-Q.xeC'Qrnp'Q"G'CP.2.(tffX) T C'R E CIQR MP pVQcOGPR Dem. [*176-14] 2D. (t'xc) T O'R e CIQR(1 F.- *116-12. *93-11.2D1-:. Hp. Se (G'P T' CR)4'C'1?. T= ('x) T CT].): (S'B'R) Q (T'B'R): (ay). yR (B'?): [*10.53] 2: (S'B'R) Q (T'IB'R):yR (B']?). y =+ B']?. 2y. S'y = T'y: [*176-19. (I)] 2) S(Q-R) T (2) F.(2). *176-16 2F:.llp.:SE CIPR. 2.S (QR) (tfX) TGCIR} (3) F-. (1). (3).2)F. Prop *261-64. F:Rd?2fin-t"A.PlessQ.2).PRlessQ-R Dern. F.*254-55.2) F: Hp.:).(HP').P'smor P. P'C-Q. a!O'IQ MP'Q"CC'P' [*261-63.*250-13] 2). (P'). Psmor P. P' CQ.! C'QR rp"QRftCt(P')R. [*176-35-22] 2). (M).Msmor PR* M C QR. a!CQ C Q'QCCM [*254-55.*261-62] 2D. PR less QR: 2 F. Prop *261-65. F: PEfIinfin.QEf~fin.2.QlessP Dem. F. *261-11-14'42.2: F: Hp.:2. C,'Pe Cls refl. C'Qe Cls induct. [*124-26] 2. Nc'C'P> Nc`C`Q. [*255-75] 2. Q less P: 2 F.- Prop

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*262. FINITE ORDINALS. Summary of *262. Finite ordinals are defined as the ordinals of finite well-ordered series; infinite ordinals are defined as the ordinals of infinite well-ordered series. In virtue of *261'42, finite ordinals are those whose members have fields which are inductive, and are also those whose members have fields which are not reflexive. Finite ordinals have the formal properties which cardinals have but which relation-numbers and ordinals in general do not have, i.e. their sums and products are commutative, and the distributive law holds in the form L (v+r) = (L v) +((X), as well as in the form (v + a) X, = (V X t) + (a * X), which was proved generally in *184'35. The distinguishing properties of finite ordinals are most readily established by means of their correspondence with inductive cardinals. In general, two well-ordered series whose fields have the same cardinal need not be ordinally similar, but when the cardinal of the fields is inductive, the two series must be ordinally similar. Hence the ordinal of a finite wellordered series is determined by the cardinal of the field of the series. We put generally rfl =rn CFt Df. The result is that, if p is an inductive cardinal, pu is the ordinal of all those series whose fields have, members. Thus there is a one-one correspondence of inductive cardinals and finite ordinals; and in virtue of this correspondence, the formal properties of finite ordinals can be deduced from those of inductive cardinals. It will be observed that, according to the definitions already given, I. 0, = n n o"A by *250-43, I. 2 = Q n CG"2 by *250-44. 9-2

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132 SERIES [PART V Hence the notations 0r, 2r are particular cases of the general notation r. But in virtue of *200'12, we have, by the definition of tr, 1. Ir = A, so that 1r does not take its place in the series of finite ordinals. Our definitions in this number are NO fin = N0r"I fin Df, NO infin = Nor"Qf infin Df, Pr, = f2 At C Df. It will be observed that for the sake of convenience we have defined NO fin and NO infin so as to exclude A. The definition of pr is chiefly useful when,t is an inductive cardinal. The number begins with various elementary propositions, partly embodying the definitions, partly concerned with tr. We have *262'12. F: P e r.. P e I1. C'P e, *262'18. F: p E NC. [! Pfr. D. p = a"Pr This proposition does not require that Pr should be a relation-number. If u is a reflexive cardinal, Pr is not a relation-number unless it is null, because series of many different relation-numbers can be made with a given cardinal number of terms. When Pu is a cardinal, "31! p'u means that classes having iA terms can be well-ordered. *26219.:., v e NC.! Ur.: =.- = = vr Thus the relation of,A to PAr is one-one so long as A is the cardinal number of a class which can be well-ordered. We next prove that if p is an inductive cardinal other than A or 1, Pr is a finite ordinal, and that every finite ordinal is of the form,,. for an appropriate p. We have *262-21. F-:, e NC induct - t1A - '1. -! ur *262'24.: P e NC induct - 'A - '1.. r e NO fin We prove this by means of an inductive proof that two series are similar if their fields are inductive and similar. *262*26.: a e NO fin. -. (t). t^ e NoC induct - 'l. a = Pr Hence we easily obtain the properties of finite ordinals from those of the correspondingZcardinals. Assuming that p, v are inductive cardinals other than 1, we have *262'33. pLi + Vr = (Fc + v), ~262-35. P-+ 1 = (u +41)r, if p + 0, *262'43., X ir = (F/ X V)r

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9 ECTION E] FINITE ORDINALS 133 *262-53. a, exp, v, =0()r, if V + 0, *262-7. 1-k > V a Ur > Mrv Hence if a, 8, r are finite ordinals, *262-6. a —/3=/0-ja *262-61. a*A3=/=*aa *262-62. a;(/3-iy)=(a*/)ia(a rj(y) *262-63. (a * /) exp. ry = (a expr ry) x (/ expr ry) Thus the arithmetic of finite ordinals obeys the same formal laws as the arithmetic of inductive cardinals. *262-01. NO fin = Nor,"Qf fin Df *262-02. NO infin = Nor"fl infin Df *262-03. I-'r f I A Df *262-1. -: a E NO fin. (HP). P e f2 fin, a = N,r'P [(*262i01)] *262'11. F aENO infin.. (HP). P ef~ infin'. a Nr'P [(*262-02)] *262'111. F:.aeNOfin.=-:aeNO:at i- a.v.a=r: a e NO a zj i -I at * v * a: = Dem. F.- *262-1.. F:. a e NO fin.:a e NO:(P). P e fl fin. a = Nr'P: [*261) 36] a:e NoO:(P): Nr'P + i -i-Nr'P. v. P =a: Nr'P [(*255-03)] a e NO:a + i a. v. a = 0: (1) [*lSO4.*155-5]=:a eNO: ta i at.v.a = Or (2) F.(I).(2).)D F. Prop *262-112.

F: aE NO infin.E a e NO -fr. t +a = a [*262-111. Transp. *261-13] *262-12. F: Pe p. =-. P e C'. CIP e p [(*262-03)] *262-13. F:Nr'P iseNO fin. P e ffin..Pe f2. C'P e Cls induct Dem. F. *262-1. F: Nr'P e NO fin.. (HQ).Q e fin. Nr'P = N,,r'Q. [*152-35. *155-13]. (HjQ). Q E fl fin. P smor Q. [*261-183] P. Pe ffin. (1) [*261-42-14]. P E f. G'P C Cls induct (2) F.(I).(2).)F. Prop

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134 SERIES [PART V *262-14. F:Nr'PeNOinfin.E-.Pefl~infin.=E.Pe61.0'PcClsrefl [Proof as in *262-13] *262-15. I-:.aeNO.):aeNOflin.=-.C"aeNCinduct Dem. F. *262-13. *120-21.)D F: N~r'PENO fin..Pe 12. N,,c'O'P eNC induct (1) F:. Nor'P eNO.):Nor'P eNO fin..N~c'C'P eNC induct. [*152-7] C "'Nr'P e NC induct (2) F. (2). *155-2.) F. Prop *262-16. F:. a eN)O.): a E NO infin C "a E NC induct.. G1a e NC refi [Proof as in *262-15] *262-17. F:P e f2.).P e(Nc'G'lP),. Dem. F. *100-3.) F. C'P e Nc'C'P (1) F. (1).*262-12..)F. Prop *262-18. F:,uNC.g!1tr.).4tCOjar Dem. F. *262-12.)F. C "P{r C/ (1) F. *262-12. F::ae u. PEa.)D. a, C'P e,u (2) F.(2). *100-5.)F:HP.a E -P e Ir.).a smCG'P. [*73-1]:). (HS). S elI-*1I.-a =D'S. C'P=PS [*151-1.*150-23]:).(AS).;PsinorP.C'S;P- a. [*2)1-111.*262-12]).(gS). S;P Ef. C'S; P= a. [*262l12.Hp]).(HS). S;PE/Ur C'S;P = a. [*37-6])a eC" ",U (3) F. (3). *10-23.)DF: Hp.). /zC C",wr, (4) F. (1). (4). DF. Prop Dem. F. *262-18.:) F: Hp v 1x) V., C"Vr [*262'18] (2) F.(1).(2). DF. Prop

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SECTION E] FINITE ORDINALS 13.5 *262-2. F-. Cls induct -1I = C1"(12 n Cnv"Iil) Dem. FI-.*261-29.:) F-. Cls induct - 1C - fl Cnvl"fl) - 1 [*200-12] - Ifln Cnv,'6P).:) F-. Prop *262-21. F:,cLENCindu~ct-t'A-t'1.:).a!p, Dem. F.*120'2.*100-43.)F: Hp.). (aa).a e * a eCis inducta, a c1. [*262-2] D.(Ha, P). a e1t.kPei7L.G'IP =a. [*262-12]:). 2! /t:)DF. Prop *262-211. I a eCis induct -1.). 2! (Nc'a)r nt~ja Dem. F.*262-21.*103-12.)F:Hp.).! (Nc'a)r. a eN,,ca. [*262-12]).(HP). P E (Ncla),r. (J'P e Noc'la. a E Nic'a. [*63-13]) (HP). PE(Nc'a),. 6C'Pet'a. [*64-24.*35-9]). (HP). PC (Neca)r. P E t'(a T a). [*64'1 1]).!(Nc'la), t,'a:) F. Prop Dern. [*110-4]. e NC -t'A (2) F. *93-103. *250,13. F: Hp.). C'P =t'B'P vU'P.B'P,-.,e 'P. [*110-63]). Nc' C'P = Nc'U'IP +01. [(1).(2)]).,L+a I= Nc'(1PP+0.I [*120-311.(1)]:)., Nc'UIP. Pe fl [*2 02 5 -5 * 2 5 0-141]., ~= Nc'fC'(P (G'P). PWUP e 2. [*262-12.*100-3. (2)] P).PG'P e p,:) F. Prop *262-213. F:. pt#0.,s11: P, Q er:)p,Q.P smor Q:): P, Q E6(P ~c 1)r. -)P,Q P smor Q Dem. F. *262'212-12. *120-124.) F: Hp-P,QE(IL +c1)r.).PtEG'P, Q~(U'QeIpL.P, QC I t'A. [*111.llp])P ~c1'Psmor Q ~G~Q. P, QC f~- fA. [*250-17]).P snior Q:) F. Prop

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136 SERI[ES [PART V *262-22. F: u e NC induct. P, Q e,r..P smor Q Dem. F.*153-101.*262-12.) F: P, Qe Or. D. P smor Q (1) F. A*200-12.)F. IrA. [*10O53] F: P,

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QC Ir.).PsmorQ (2) F.- *153-202. D F: P, QE 2,.. D. P smorQ (3) F. (2). (3). *2 ~02. D F:. c = 0. v. P = I: P, QE6ILLr.)pQa P smor Q::): P,Q e1 (+c, I)r.)p,Q. P smor Q (4) F.(4). *262-213.: F:.P, Q ECr.p,Q.PsmorQ:):.P,Q E(/~+,1I)r.:pQ. PsmorQ (5) F. (5). (1). Induct. D F. Prop *262-23. F:.P,Qef fin.):O9'PsrnC'Q.. PsmorQ Dem. F. *262-17K13. D F: Hp. C'P sm C'Q.). F, Q e (NT'C'P)r. Nc'CP E NC induct. [*262-22]). Psmor Q (1) F.(1).*151-18.:) F. Prop The above is the fundamental proposition in the theory of finite ordinals, since it enables us to reduce relations among finite ordinals to relations among the corresponding cardinals. *c262-24. F:p eNC induct-t LA-t Ll..)./-LreNO fin Dem. F. i*262-21. DF: Hp. D j!-U r (1) F. *262-22. D F: Hp. Pepu7.).,Pr C Nr'P (2) F.*262-12. *15~18.) F: P e Pr -). Nr'P C Pr (3) F. (2). (3). F: Hp. P e tr D..r = Nr'P (4) F -(1). (4). D: Hp. D ~ Pr NR - t"A(5 F. *262K12. D F: Hp. P e P.r.G CP e Cls induct. [*262-13. (4).(5)] w.1)r e NO fin (6) F. (1). (6). D F. Prop *262-241. F:. eNCinduct.Pef~.): pr=rNr'P.=-.1t=Nc'C'P Dem. F.*100-3. F: Hp.1IL = NcIP'P. D. C'P ea. [*262-12]). P /Lr a [*152-45.*262-24]. r = Nr'P (1) F. *1523. *262-18.))FHp r Nr'P.:)./ C"'Nr'P. [3152-7]).,u=NcP'P (2) F. (1).(2).) F. Prop

 *262'25. F:(a1t). queNCinduct-tf1l-t'A.a ----tr.=.aENO fin Dem. FI: a e NO fin. (rIP). P e il fin. a =Nr'P.- NeC''P E NC induct.[v262-241]. (HP). P e ffin. aNr'P.(NcCP'P)r=Nr'P. NcP'P e NC induct. [*13-172]).(H[P). a (NC'C' UP)r. NcP'CP E NC i nducet.[*200 12. *2 62 1.*1 5 51 3] 1~. p N C induct - tl- t A. a = u,tt 1 F.*264-24.:)F:(Q[jt). peNCiuduict-t'l- t'A.a=/tr.:).aE6NOfin (2) F.- (1). (2).) F. Prop *262-26. F:ae NO fin E(,)1e NC induct - t'. a= [*262,25. *103-13'34] *262-27. F: a,/3lENOfin.).a-j-1&eNOfln Dem. F.*180-21.:)F:Hp.Pea.QeI3.:).P+Qea-i-18(1 F.*251-24.:)F:Hp.:). ai-g,& NO (2) F. *18O-111.:) F: Hp (1). D. NcW'6(P + Q) =Nc'(C'P + C'Q) [*1 1O.3] Nc'C'P +, Nc'G'Q (3) F. *262-13.) F:Hp (1).:).NcW'P, NcC'QE NC induct. [*12O-45]:). Nc'C'P +,, Nc'G'Q e NC induct (4) F.(1).(2).*155i'26. *251-122. D F: Hp (1).).D P + Q6 e l. a -i-f = N,,r'(P + Q) (5) F (3). (4). D F:Hp (1).D. C'(P +Q) eCls induct (6) F. *262-1. *155-13. D F: Hp. D.! a.3H!fl (8) F.(7).(8). DF. Prop *262-271. F: a,/3EcNO fin.)D. a;/8e NO fin [Proof as in *262-27, using *184-12. *166-12. *251P55. *120-5] *262-272. F: a,/e NO fin. D. aexp,.3e NO fin [Proof as in *262-27, using *186-1. *176-14. *261P62. *120052] Dern. F. *180-2.) F:. Z e,ar-ivr.:(HP, Q).jtr= Nor'P.Vr= Nor'Q.Z smor (P +Q) (1) [*180- 11.* 51-18]):(aP Q) = N~r'P. v,= N,,r'Q. C'Z sm (C'P + C'Q): [*155-12]:(21P, Q) P e /sr.Q e6V.C'Z Sm(C'P +G'Q): [*262-12] D: (HP, Q). C'PEFL. C'Qe6v. G'Z sin (U'P +7Q): [* 10'21]:): Hp. D.O'ZEcu +-0v (2)

 F. (1). *262-12. *155'12.)D F:ZcpE /4 D~. (HP?, Q). P, QC fl. Z smor (P + Q). [*251P25a.*180-11-12.(*180-01)]:). Ze &1(3 *262-32. F:yL, vENCinduct.PE~r T.QEVr a). P+ QC tI.IVT Dem. IF. *200-12. *262-12.) F-:Hp.)

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Av-t'l -t'A. [*262-24].) Atr, vr e NO.a [*180-21] D.P-iQECtr-~VT:)I.Prop *262-33. I-: F'P vNCinduct-tf1l -. Ar+ Vr-(At+o V)r Dem. F. *262-12.)FI-:. A=A. v. V= A:) =. v *VT= A: F.*110-4.):AtAVVA)A+VA [*262-12]). (+c Or =A (2) F-. *262-32. DIF: Hp P e6tr.Q EVr).P~+Q e6 r-i-Vr. (3) [*180-42.*15-2-45]) Pr AtT VrT Nr'(P +Q) (4) F-. (3). *262-31.:) F: Hp (3). P.+ Q E (P +cV)rT [*120-45.*262-24]).P + Q E (P- +c VOr (/tk ~0 V)r C NR. [*152-45]).(At +o V), = Nr'f(P + Q) (5) F.(4). (5).*10-23. *262-21.)DF:Hp.a! P -! V.)A r -j-Vr =(At+cVOr (6) F. (1).(2). (6).) F-. Prop The above proposition still holds (as we shall now prove) when one of At and V is equal to 1, but not both. When both are equal to 1, tr+Vr =A, while (At +c VOr = 2r, Dem. F.*181L2.) F:.ZE~tr-i-=.: (P,X).~ar=N,,r'P.Zsmor(P-i~x) (1 F. *181-6.*152-7.)F:f!P.:).Nc'C'(P-4x)=Nc'C'P~01 (2) F:. Hp.): Z E Pr4). (HP). Atr Nr'P. Nc,'6CZ = Nc' C'P-,1 [*262-241-12]) (HP) AtPr Nor'P.Nc'C'0Z =A-+,a1I F. (1).*262-12. *155 1 2.)F:Z 1t6iIr (HP).P 6fl - PrN,,r'P. [*251-1132].ATlCNO. F. (3). (4). *262-12. D F:. Hp.:): Zetp-I-i.). Ze (a +, 1).:.:)F. Prop

SECTION E] FIN ITE ORDINALS 139 *262-341. F:,aeNCinduct.PeFr.:).P4*>xepr-~iDem. F.*200-12.*262-12.) F:Hp.).,E -t'1-tffA. [*262-24] D.,/, rNO. [*181-21]).P-j+xeP,1-~i-:)F.Prop *262-35. F:,aecNCinduct-t'0-t'1.:).g,-~ir (P +c1I)r Dem. F.*262-12.)F:p A..P~r= A F.-*110-4. FPA aI=A [*262-12] D.(P +e)r = A (2) F.-*262-341.)I-:Hp. P e/Lr). P-I+ x eLji- (3) [*181P42.*152-45]) /L+ =Nr'(P -j+x) (4) F.(3). *262-34.s:)I: Hp. PeP, ar).PI4+WE (j c)r. [*I120-45.*262-24] D. P 4+ x E (it + 1)r.(L +0 1)r E NR. [*1 52-451]:).(Ft +cl)r =Nr'(PI+x) (5) [*262-21] ~ p.2 P.(0 ~ (6) F-. (1). (2). (6).) F. Prop *262-36. F:AteNCinduct-tfO-ffl.).i-~Lr=(1+cIu)r [Proof as in *262-35, by means of analogues of *262-34-341] *262-41. F: p,v eNC.).ia r;vr C (,'XcvOr [Proof as in *262-31, using *184t15. *113-21] *262-42. F: LL,Pe NC induct.-P6Pr LT.Q EVr -D.P XQ E/Pr *Vr [Proof as in *262-32, using *184-12] *262-43. F: p,v cNC induct-16 L'D -./.trXVr =(/LXc V)r [Proof as in *262-33, using *184'11. *113-204. *184-15. *120-5] *262-51. F:P E NC. iv e NC induct.) r~expr. Ir C @Lv)r Dern. F.*186-5.:)F:tr,,vreNoR.vt0. RePreXPr Vr.). G'R(C"/Ur)e"'v' (1) F.*186-11.)DF:R et,aeXpr Vr a D a! Iwr. a! V, (2) F. (1). (2). *262-18. DF:Hp. v +0. R ejarexp, Vr U.CR eav (3) [(2).*251'1. *186-11]) F: RE~rexpr 1'r -D).tr NO (4) F. *262-24.:)F:Hp. v 1I. v +A. D. vr NO fin (5) F.(2). (4). (5). *261-62.) F: Hp. v I1.R 6tr exprVr.)R ef1 (6) F. (2). *200-12. F: Re a,.exPr vr.).DvtI (7) F. (3). (6). (7). F:. Hp.:e B tr eXpr Vr.) RE 6.2 O'R C IL". [*262-12] D R e (P,LV:. DF.Prop

140 SERIES [PART V *262'52. F:/,veNCinduct.Pe,Ltr. Qevr.).(PexpQ),E(p, aexprv,) Dem. F *200-12. *262-12.:) I-: Hp.).v e - tf1 - I(A. [*262-24] r)q tr vENO. [*186-13.*152-45])(P exp 9) E ('r eXPr vr):) - Prop *262-53. F:,s, veNCinduct-tff1:.arx,-V (z) Dem. F.*262-12.*186-11.:)F:.p&=A.v.v=A:). PrexPrVr=A (1) F.*116-204.*262-12.) F:.P= A. v. v=A::).(pvr=A (2) F.*262-52.)

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F:Hp.Pe6 r. Q eVr).(PeCXPQ)6 (preXPr Vr) (3) [*186-13.*152-45]:). Nr'(P exp Q) = Ftr eXPr Vr (4) F.(5). *120-52. D F:Hp (3)..). teeNC inducet (6) F. (5). F:H 3.:f! (-) [*200-12.*262K12]) p. +~ Ij (7) F.(6). (7). *2 6 224. F:llp.).(.v)rE NO (8) F. (5). (8). *152 45.) F: Hp (3).).Nr'(P ex p Q) =(14)r. [(4)]) r. (X ePr ir = (119r (9) F(1) (2). (10).) F. Prop We are now in a position to establish the commutative property of addition and multiplication of finite cordinals. This is effected by means of *26'233 and *262-43. *262-6. F:a,/3,ENOfin.).ai-/3=/3-ia De~m. F. *262-26.:)F:Hlp.). (Hp,). i' NC induct -t ff.a =pr-'R* = i'r [*16-32]).(a1a,v)./,v, veNCindiuct-tc1l.a-i,8-(r+,vra=p~r./3=Vr. [*2623351]). (Hpt, v). p, v e NC induct - tf1. a 4~, / (v +,, V)r. a= 11r.1 =3 Vr. [*262-33]). (E[/&,v).1Lt, veNCinduct-tcl. a /38Vr-~p4r a=/tr./3Vr. *262-61. F:a,/83eNOfin.D.a*/3,=/3*,a [Proof as in *262-6, using *262-43 and *113-27] *262-62. Fa/,e~i..*/ —y=a/)-ay Dem. F.*262-27-61.)F:Hp. D.a*Q(3-i-y)=(/3i-y)*(a [*184-35] (/3*;<a)i- (7 y a) [*262-61] =(a */3) -i-(a y): F.Prop

SECTION E] FINITE ORDINALS 141 *262-63. F:a,,3,'yENOfin.).(ca;fl)exp,.y= (aexp,.'y)*<(flexpry) Dem. F-.*262-26.) F. *262-43.) F.*113205.)F:Hp(2.)=.:)*x UveNVindc (6) F. (5). (6). F223) Fp:2) Hp (2) v Or) 1 /~X ~ xr~.={/~Xpt F[- *1207631] p 2) x). v Ntndc (8) I-.(5).(9)6 F:p).*225.)F:p(2.,+r-1 (10,xp~~jP,,,) W Similarly X.Hp7).)v~r (11) F. (10). 652). *12052p *26243.+ Or F: HP (7. p>X ~) p) [*117631] =A (/h exrVr (rep (12) F.(2).(7).(12).).Dp'+I(I0 FiiaryF: Hp (7).) #). r)ep v1r = (Itrep r)*(rxp (I13) F.(10).(11).*126-"29.*224.)H()D.P',Vur();(F: Hp 'y +Or). (a *,k,) exprY = (a eXpr7y)*(expr'y) (14) F.*186-2. *184-16. FHp.7'Y= Or.-)(ac,3) expr'y= Or.(a eXpr'y)*,~(Iexpr'y) =O (15) F.(14).(15).). Prop *262-64. F:axeNOfin.:).a-0=-i-ca Dem. F.*262-35'36-26.*110O51.)F:Hp. a#Or.).a-i-=ti-a (1) F.*161-2201.)F:a=Or.:).a-+1=Or.1+ca=Or (2) F. (1). (2).:) F. Prop *262-65. F a,/,E NO fin./3# Or)a;(1 i) = (a 18)+ct Demn. F. *262-61.) F: Hp.). a *~ (,8+ 1)=(i) * a [*184-41] =(3a)i-a [*262i61] =(~I —:F.Prop *262-66.FafeNfn3~r)*(-1)=i(j) [Proof as in *262-65]

142 SERIES [PART V Dern. F-.*262-21. *117,12.)D F: Hp..p > v)a! ur -a! vr [*255176.*262-24] PFr > Vr (2) F.(1). D)F:Hp -uk< V.).Ditr < Vr (4) F. *262-21. DF: Hp. sm"v D.).(aP). Ik=NocC'CP./Aksm "v. [*103-4] D. (HP). p= Nc'C'P. v = N cP'PC. [*262-241]:). (HP). j~= Nor'P. vr = Nr'P. [*1a554]:).,Ur= smorttv (5) F. (4). (5). *117-104.)DF:Hp.IL~<v.)D.F r — 'Vr (6) F-.(2). (7).)D F. Prop *262171. F:aENOfin-t'Or.:).(a13).flENOfin-t'Orvt'i-a=1+i Demn. F.*26211.*261-24.)FDlHp.) DK a! a ^ '(B ICflv) (1) F-. (1). *204'483. (*181-04).) DF. Prop *262-8. F: a,13eN0.,yeNOfin.a<E13.:).aeXPrrY<IexPrrY [*261P64] *262-81. -: a,/3,eNO. rycNO fin. aeXPrrY=I8cXPrY.:).a=sinor"13r, Dern. F. *262-8. Transp. *255-42. D F-: Hp. D) (a < p3). —,(a > /3). [*255-112] D. a = smor"/38 D F. Prop *262-82. F:aeN`0fin./3,EN`Oinfin.:).a<13 [*261-65] *262-83. F: aeN0O tOr.i/0/,yE NO

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finf< y.)Da eXprI3 < aeXPr~Y Dem. F. *255-33.) F:. Hp.:): (aw). eNO -t'Or. -Y =18 V./3t+r -Y/3+8 (1) F. *254-51. D F: Q C P.)D. - (P less Q) (2) F.(3). *262.82. Transp.)DF:Hp.y = f3+ D.). e N finl (4) F. *186-14.) F: Hp (4).,w +Ora1=0r3Or.).a eXpr'y= (a eXpr/3)(a exPr) (5) F. *262-71-272.) F: Hp (5). D. (f8). 8 e NR -tfOr v t'1. a exPr 3=~ 11 [(a5).(4).*255-573]:). a exPr fY > a eXPr /3 (6) F.*255-51.)DF:Hp (4) +#Or -/=Or D.)a exPr F7> aeXPr/ (7) F. *18622. D F: Hp.- 830. Or -Y= /31-1.). a exp, ry =(a exPr/8);(3 [*262-71.*255-573] D. aeXP~r Y > a exPr/3 (8) F. (1). (6). (7). (8).)3 F: Hp.). a eXPr rY > a eXPr /:) F. Prop *262-84. F:PeQ2-tfA. Q,Ref2fin.QlessR.).PQlessPB [*262-83]

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*263. PROGRESSIONS. Summary of *263. If R is a progression in the sense defined in *122, i.e. a one-one relation whose field is the posterity of its first term, then Ro is a serial relation, and the series generated by Ro is of the type which Cantor calls co, i.e. the smallest of infinite series. It is easy to prove that all progressions are ordinally similar, and that, if all inductive cardinals exist, the series of inductive cardinals in order of magnitude is of the type ac. Thus so is an ordinal number, which is not null if the axiom of infinity holds. Most of the properties of co are easily deduced from the corresponding properties of "Prog," which have been proved in *122. The definition is w = P {[(R) R e Prog. P= Ro} Df. The axiom of infinity implies that " less to greater" with its field confined to inductive cardinals is a member of co, or, what comes to the same thing but is easier to prove, that {(NCinduct) 1(+c l)},p is a member of ao (*263'12). Thus the axiom of infinity for the type of x implies the existence of) in the type t3'x (*26.3132); and generally the existence of co in any type of relations is equivalent to the existence of No in the type of their fields (*263'131), because 0N = D"co = C"co (*263'101). By using the fact that in a progression R (in the sense of *122) all the terms are values of VR, where every inductive cardinal occurs as a value of v (which was proved in *122), we easily deduce that if there are progressions they are the series that are ordinally similar to the series of inductive cardinals (*263'161). Hence both "Prog" and co are relation-numbers (*263'162'19). Moreover, by *122'21'23, co consists of well-ordered series (*263'11). Hence o is an ordinal number (*263'2). We next prove that progressions are infinite series (*263'23), and that a series contained in a progression is finite if it has a maximum (*263'27), and is a progression if it has no maximum (*263'26). It follows that, assuming the existence of progressions or the axiom of infinity, co is the smallest ordinal which is greater than all the finite ordinals (*263'31'32). Connected with this is the fact that the predecessors of any term in a progression are an inductive class (*263*412).

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144 SERIES [PART V *263'44'48 give various formulae for w), any one of which might be taken as the definition. We have *26344. F. o = n - A n P ((IPI = ('P. E!

B'P) I.e. progressions are existent well-ordered series in which every term except the first has an immediate predecessor, and there is no last term. *263'46.. o = n P (E! B'P1. E! B'P) I.e. progressions are well-ordered series in which there is only one term having an immediate successor but no immediate predecessor, and there is no last term. *263'47. F. o= P {a C C'P.)a: a e Cls induct..! C'P n p'Pa} I. e. a progression is a well-ordered series in which any sub-class a stops short of some point of the series if a is inductive, but not otherwise. This proposition will be useful in the next section. *263-49. F. f finu o = n P ('P, ('= P)= i n P(P= Pfn) I.e. finite well-ordered series and progressions together are those wellordered series in which every term except the first has an immediate predecessor, and are also those in which every interval is an inductive class. From *261'45 it follows that, if P is an infinite well-ordered series, P confined to the terms at a finite distance from B'P is a progression, i.e. 4 -*263-5.: P e 2 infin. D. P ('B'P v Pfn'B'P) e o Hence it follows at once that an infinite ordinal is at least as great as w, and therefore infinite ordinals other than o are greater than co, i.e. *263'54. F: a NO infin - t'w. a. a> co The remaining propositions of this number are occupied in proving ow X 2,= o (*263'63) and o a = co if a is finite and not zero (*263'66). It is not the case that 2, X co = co or a X co = o. Cantor has varied his definitions of multiplication as regards the order of the factors. Originally, he adopted the same rule as we have adopted, but in later works he inverted the rule, so that what we call 2r Xko he calls co X 2r, and vice versa. Thus with his definitions in his later works, 2, X o = o but co X 2r o. We have reverted to his earlier practice, for various reasons, but chiefly in order to have Nr'I'(P Q) = Nr'P Nr'Q (cf. *172). Whichever rule we adopt, there are some inconveniences, so that the question as to which is chosen is not of great importance.

SECTION E] PROGRESSIONSj 145 *263-01. o=PI(R).RE.Prog.P= R,,j Df *263-02. N = J A [e NC induct. v = (~ +,, 1) n to'1. Dft [*263] The above temporary definition of N is the same as that in *123. *263-1. I- P E o. (a.(1R). R E Prog,. P= R [(*263-01)] *263-101. F. No= D"Co = C"o [*123-1. *122-141. *91P504] *263-11. F. (0 C f1 Dent. F.*122-23-141.*2f63-1.DF:PPecocraCCP P H!a.:) E! minpla (1) F. (1). *250-125.:)I. Prop *263-12.:Infin ax.). NpE,& [*123-25.*263-1] *263-13. F:a[!No(x).= - co n t"'x Dem. F. *263-101. (*6502).) I- a! i (x). * (a.). P) co. G'P E t't'x. [*64-57.*63'5]zu-.(aP).PEw.Petil'x:) I.Prop *263-131. I:a! (KN),.r a'w tcfa [Proof as in *263 13] *263-132. F: Infin ax (x) = a co A t33%x. Dem. F. *125-23. *263-13.: Infin ax (x). Co. n t'tt2'X. [(*64-011P014)] -.! A t33'X:) F-. Prop This proposition asserts that, if the number of individuals of the same type as x is not an inductive number, then there is a progression whose terms are of the type of tZX. This progression will be that of the inductive cardinals which are applicable to classes whose terms are of the same type as x. *263-14. F:ReProg.P=R?,.).P=Pfn=Rfn.R=PI Dem. Fa*121-254.)IF: Hp. P, = R,. [*121231.*122-1-16]. PI=1. (1) [Hp]. (P1)P0 = P. [*260-27.*263-11]). PM=P. (2) F. (1). (2.= (3) F Prop R. & W. 111. 10

0 146 SERIES [PART V *263-141. I-: P e w.). P1" E~ Prog,. P = (P,)f. = (P,)~,0 Dem. F. *263-1.)F H p. D. (HJR). R E Prog. P = B1O [*263-14] D. (gR). Re Prog. PI=R. P =Rfl P = Rp~. [*13-195-]).'P P Prog. P = (Pi)f,, = (P1)P.:) F. Pro p The above proposition shows that every interval P (x ~- y) in a progression is an inductive class. *263-142. i-: R, SEProg. RPO=8SPO. Y. R= S Dem. F. *263'14.) F: Hp. D. -R = (P) [*263-14] =S:)D F. Prop *263-143. F:P, QE w.P1= Q.D. P= Q Dern. F. *263-1 F) FHp.).(JR, 8). R, S e Prog. P = R,. Q = Sp.. PI Q1. [*263-14]).(:fR, S). R, SEProg. P= Rpo. Q = SPO R =PI.S = QPI =Q1. [*13-17]).(aR,S). R,SeProg.P=.R1O. Q=Sr,0.R=S. [*13-17]).P=Q:)F.Prop, *263415. F:RcProg. S=`fi veN`Cinduct.x = (V +c1)RI..S sm N Dem. F.*123-3.:)F:Hp.:).Sel-+1 I. D'S=D'R.PIS=NCinduict (1) F. *123'21.:) F. NC induct = C'IN (2) F.*1 10'56643. DF Hp. (p,D +1) N(v ~0,1). D.v+I=,O+02 (3) x (S;NY) y.. (Eprt). At e NC induct. X = (,U +,0 l)R. y =(+02)RB [*121-332-131] (Hp~).,u c Nd induct. (B'R) R, x. (B'1R) (R,. R)y. [*122.341.*121-342] E. xRy (4) F. (1). (2). (4). D F. Prop *263'151. F:Re Prog.D.BRsmor N [*263-15] *263'152. F: R eProg. Qsmnor R.). Q eProg [*123-32] *263416. F:1? eProg.D). Prog =NrR = N~r'N [*263-151-152] *263-161. F:! Prog. D. Prog = Nr'IN [*263-16] *263-162. F. Prog e NR[*611. *142] [*263-161. *154-242]

SECTION E] PROGRESSIONS 147 *263-17. F:Pew.).o=Nr'lP=Nr'N1, Dem. F. *263~11.:)F:Hp.:.(HR)..Re Frog.P= =R,,. [*263-151] D. (R). R smor N. P = RP. [*151P56]). P smor Np.. (1) [*152-321] D. Nr'P = Nr'NP0 (2) F. *151P59.): P e c. Q smor P.). Q smor P,. [*263-141-152] D. Q, e Prog (3) F. *150-83. D F: P E o. S e Q smor P..,), = 8;(P,)po [*263-141] = S;P [*151 11] =Q (4) F. (3).(4). c*263-1. D)F: P e co. Q smor P.)3. Q e w (5) F-.(1). DF: P, Q eco.D. P smor Q (6) F. (5).(6).) F: Pe w.).w = Nr'P (7) F.(7).(2).)D F.Prop *263418. F: a! o. D. = Nr,'Np [*263-17] *263-19. F. oe NR [*263'18. *154-242] *263-2. F. o e NO [*263,1911. *256-54] *263-22. F: P e w.:. GY C D'P.r E! B'P. E! B'P [*122-141. *2631. *122-11] *263-23. F. o C fIl infin Dem. F.*261'35.Transp.*263-11P22.)F: Peco.).C'P,e eClsinduct-t'A (1) F. *263-22. F: PE co.). g! G'P (2) F. (1). (2). D F: P E o.). G'P e Cls induct. [*261-14-41.*26311]). P e fl infin: D F. Prop *263-24. F: a! co.).w O E NO infin [*262'14. *263'17-23] *263-26. F:Peco.a!a,'C'P.r-E! maxp'a.).P aeo Dem. F. *263-1. *205123.) F:Hp.. D.(HER).R Re rog.P= =R,O.: i Qa" iC R atCn CRCR.6'a. [*122A442-45] D. (a?). R e Prog. P = R,0. P C a -!(P C a) 2 e Prog. [*263-1]). P aeo::)F.Prop *263-27. F: Peo. E!maxp'a. P ae &2 fin Dem. F. *122-43.*263-1.)F:Hp.:)..a n C'P e Cls induct. [*37A41.*120-481]:). C'(P C a) e Cls induct (1) F. *263-11. *250-141. F: Hp.:). P C ae f2 (2) F. (1). (2). *261-14-42.) F. Prop 10-2

148s SERIES [PART V *263-28. I- P e &.). Ser n Rl'P Cco v ~ fin [*204-421. *263-26'27] *263-29. F: Pe w. Qefin. D. Q less P [*261P65.*263-23] *263-3. F: P e Co. D. less'P = fl fin Derm. F. *254-1..*263-17. D F: P e (o. Q less P.. q! Nr'Q ^ Rl'"P. Q, E o. c2 e 1 [*263-17].([R). Re Nr'Q n Rl'P. Rr, e co. [*26328]). (HR). R E Nr'Q n I fin. [*~261-183] 3. Q6 f fill (1) F.(1). *263-29.) F.Prop *263-31. F:.a! wo.):a<w.E=.aeNOfin Dem. F. *25517. *263-17.) D-:. P E wo. D: Nr'Q < co Q. Q less P. [*263-3].QE f fin. [*262-13] -. Nr'"QENO fin: [*152-4] D: a e NR. a<.to. n. a e NR. aENO fin: [*255'12.*262-1.*152-4]): a < o. E. a E NO fin:. D F. Prop *263-32. F:. Infin ax.): a < o.-a * a E NO fin [*263-31-12] *263-33. F:a<wo.). aeNOfin Dem. F. *2551.*155'13. DF:Hp.).a!o (1) F.(1).*263-31.) F.Prop *263-34. F.ti-co=o= Dem. F.*262,112. *263-24. F:Hp.!w.coo. ico=co (1) F.*181L4.)F: co=A.A.1iD. co=A (2) F.(1).(2).:)F.Prop *263-35. F:aeNOfin.).a — co=w Dem. F. *180-61. *263-18. D F-: a! w.. 0, co = c (1) F. *180-4. DF:co=A.A.OOri-w=A (2) F.(1.(2.) Or CO C (3) F. *18157. *263-34.:) F. 2, c = 1 + w [*263-34] =C (4) F. *262-36. D F: E NC induct - tf0 - tf1. D. (p +1,)), + Co = Pr 1 t c [*263-34. *181~5~] = E5f + o (5) F.(5).) F: Le NC induct - ffO - ff1. wrio=co.). (P + 1) rtco + o (6) F. ()4. (6). 1nduct)F:,u eNC induct - t6O - II.)r(CDo = O (7) F*26226])F: P e NC induct - til D- o r FCO = p: [*~262-26] D F: a e NO fin. D. a + w = co: D F. Prop

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SECTION E] PROGRESSIONS 149 *263-4. F: P cco.:). D'PC f2fin. Nr"D'Pv =NO fin Dem. F. *254-182.:) F: lip.:). D'Ps C less'p. [*263-3] D.DRIP C f2 fin (1) F. *26.3'31. DF:.Hp.:):cL<Nr'lP.E=-aEN~fln: [*256-11]:a e Nrl"D'Ps,. E*a e NO fin (2) F-. (1). (2). D F. Prop *263-401. F:PEw.aiEsect'P-tfA-tfC'P.).E!maxp'la Dem. F. *250-65.)FI:lp.).P ~ae,- eeNr'P. [*263-17].P a' o. [*263-26.Transp].E! maxp'a:) F-. Prop *263-402. F: Pcw.:). sect'P - tA-tfC'P= P*"6C'P Dem. F-. *205-13122. *26.'1401.:) F: Hp. a E sect'P - tA- t'C'P. D. a v P'la = P'r-naxp'a v t 'inaxp'a. [*211-1.*91-54]).a = P*' rnyaxp'a. [*205-111])a cP*"'6cP (1) F. *211-313.)F. P*6"C'P C sect'P (2) F. *90-12.)F.P*",C'P C- t'A (3) F. *20,5197.)F:Hp. x cU'P. D. E!raxp'P*'x. [*263-22] D. P*'xtG'cP (4) F. (2). (3). (4). D F: Hp..P*'CP'PC sect'P - A - t'IC'P (5) F. (1). (5). DF. Prop *263-41. F:Peco.).Ps ~D'Ps=PVP*;P Dem. F. *213-11-141 1 51.) F:. Hp.): Q(Ps ~D'P,)R.. (gaj3). a,j8esect'P-t'A-t'C'P.aC,8.a +f8. [*263-402] (ajx,y). x,y E C'P. P*'x C P*'y. P*'x tP*'y. Q = P P*'lx. R P ~ P*'y. [*200-391] EE. (ax~, y). x, ye 'P..P*'x CP*'ly. w +Y. Q =P ~ P*'x. R =P ~ P*'y. [*204-32.*90-12] E.(ax,y).aP~/.w Y. P*jXCP*'y. Q=P~ P*'x. R= P P*'y.

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150 SERIES [PART V [*201-1415]u.(axe~). xP*y -X +y.-Q = P P*'x.R = P P*'Y. [*201-18].(ax,y).wPy. Q = P P*'(X. R= P P*'y. [*150-1]. Q(P r;P;P R:. 21F. Prop *263-411. F-: PEWc.:). C"'D'Ps = v (((f L 'A Dem. F -.*213-141. *263-402. 2 F: Hp.:2. C"D'"Ps C~CC*f [*93-103] = C(rCPCfCpv if (J'P P*'B'P [*201P521.*202-55] = P*"P(IP v tf C'P P*'B'P [*201P521.*200-35] = P*"G'(IP u tfA: 2 1-. Prop

263-412. F-: P Eco. 2. P'Ix, P'w e Cls induct Dem. 1-.*205-197.21-F: Hp. x eG'P.). E! maxp'P*'x. [*263-27.*202-55.*120-213] 2D. P*'w E Cis induct. (1) [*120-481] 2. P'~x e Cis induct (2) F-. (1). (2). 2 1-. Prop *263-42. 1-:PEos.. sgm'lP=Aj~(C'lP) Dem. 1-. *212-21. *211-12. 2 1-. (1). *211-1. *205-123. 2 F-: Hp. a (sgm'P),3. 2. a,8 fiE sect'P.-, E! maxp'la. -E! maxp",. [*263-401]). a,/3 e tA v tfCcP (2) 1-.*37-29. 21-:a=A.2.a=P'la (4) F-.*263-22.2D1-: Hp.jL3= C'P.2D.43= P"$, (5) 1-. (3). (6). 2 1-. Prop *263-43. 1-:Peo.2).GPP1=QP Dem. F-. *263-141.21-: FHp. 2. U'P =(I(, [*91-504] -PGP, F1. Prop

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SECTION E] PROGRESSIONS 151 *263-431. FPf-~AQP='.EBP).ea Dem. F. *261P35.Transp.)F:Hp.).P EllIinfin. [*261-44]).P1 Pf,'B'P E Prog. [*261P212]). PI P'B'P e Prog. [*202-524]).P1 E Prog F. *261P212. DF p.P=(Ip F (1). (2). *263-1)F.Prop *263-44. F. 2S - tn P"'((I'P, IP. E! B'P) [*263-4322-431] (1) (2) *263-45. F-co-fl-tfAu'P(P=Pffl.,-~E!B'P) [*261P212. *263-44] *263-46. F o = f2 nP (E! B'P1.,,E! B'P) Dem. F. *1214305. *93-101.)3 F: PeQ.E! B'P.U 'Pi +U'P g.!U'IP - U'1P1.U'1P DIP - fB'P. [*250-21]) a! DIP,. (UP1 - t'B'P. [*93-101].! BITP1-t'B'P (F.*1 21305. *2-5021.)'F:Pe I- t'A.). B'Pe B'P1 (2 [*53-3] E! BE!B'P1 F. (3). Transp.) F: P e SI. E! B'P1.,' E! B'P.). (IPP1 = (L'1P [*263-44] D. Pe (F. *250-21. *263-44. DF::P cOw.). B'P = B'P. [*250-13])E! BcPj F. (5). *263-44. D F: Pe c.D. E! B'P,.r ---E! B'P F. (4). (6).) F. Prop 1) 1) 3) t) 5) *263-47. F.w=Sfv'PtaCO', P.)a,:aeClsinduct.=.2j!C'Pnp'P"aI Dem. F.*254-52.)F:P EcO*aCC (I'P. j!C('P np,'Plla.).(P ~a) less P. [*263'3] D. P ~ a e SI fin. [*261-42-14]:). CY(P ~a) eCls induct. [*202-55.*120-213] D a E Cis induct (F. *261P26.:) F: PE e) a C C'P. a E Cis induct. 2[! a.:). E! maxp'la. [*263-22] D. aj I P'maxpl'a. [*205-65.*40-69] D). a! C'Pnip'P(P a (2) 0)

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152 SERIES [PART V F:. Pec. aC C'P.:):caeClsinduct.! C'P np'P"'a (3) F-. *40-2. *120-212.)3 4 -F:: PeQ:. aC CP.~ae Cls induct.Ea.!WP np'P~a:. D4qiP (4) F. (4). *200'51.) F:Hp (4). D. C'PrE Cis induct (5) F. *250-16.) F: Hp (4). a[! U'IP - PP".). P'min p,(cI'P - G',P,) E Cis ind uct. [*261-26] D. E! maxp'P'rninp'(G'lP - U'PO) [*205-252] D. minp'(PIP - WP1I) E PP (6) F. (6). Transp. D-: Hp (4).) WUP =(lpP (7) F.(4). (7). (8).)F:Hp (4).). Pe co (9) F.(3). (9).)DF.Prop *263-48. F~ofn lCI.),:~e~rf.-,aCPp [*263-47. *261-47] *263-49. F.~flfnvuo=fl1P ((IPP1=G'P)=flrtP(P=Pfl) Dem. F.*261-22.*263-44.)F:Pef~finvw,.:).([PP1=:PP (1) F.*261-34.*263-44.:)F:Pefl.(IPP=U',P.).Pef~finvwo (2) F.(l). (2).)F. f2fin v w = f2 n P (PP 1=PP) [*261'212] = n P(P=Pf.).)F. Prop.*263-491. F:Pe&2finvo.:).P= (P1),,0.P,=(P1),, Dem. F. *263-49. *261P212. D F:Hp. D.P=(P1)~0.(1 [*91P602. *121-103] P(~Y I( ) [*121-11])Po =(P1)" (2) F. (1). (2). DF. Prop *263-5. F: P e f2 infin. P ~ (t'B'P u Pf1,'B'P) E w Dern. F. *261-45.)F: Hip.:)..PIPf.'B'P eProg (1) F. *260-33-27.) F: lip.:). (P1 rPfn'B'P)po = Pfn ~ (t'B'P v Pfn,'B'P) [*260-32] = P ~ (if B'P v Pf,,'B'P) (2)

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SECTION E] PROGRESSIONS 153 *263'51. F:Peflinfin.). Dem. t'(ffBP v Pfn'B'P e D'r(Pe A% I). tfB'IP v Pfn,'B'P E P lsgm'P I -.*263'5-22. I-F Hp.:).,, E! maxp'(ffB'P U Pfnl'B'P)(1 F.*260-11.:)F:Hp.yetI,'P-Pfn'B'P-XePfn'B'P.). P (B'P ~~ y) e Cis induct. P (B'P i — x) E Cis induct.[117-222.Transp])D. (yPx) (2) 4- -*) 4 -F M.(2. Transp.) F:Hp.D). P"IPfn1'B'P C B'PUPfn'B'P (13) F. (3).- *93-101.: F: Hp.:). P"I (t'B'P C Pfnl'B'P) C L'BPVPnB'P v 4 -F.(I).(4). *211'41.DF: Hp.D. t'B'PVUPfn'B'P eD'(Pe tI). (5) [*212-152] D. tfB'P v Pfn'B'P e U'~sgm'P (6) F. (5). (6). D F. Prop *263-52. F:Pef~infin-w.:).(ax).xeGI'P.Pfn'B'Pvt,'B,'P=P'x Dent. F. *263-49. Transp. D F: Hp. D.:! P'P - (IPP. [*260-27]) IC!UP - Pfnl'B'P. [*250-121]).E m ui np'(PIP - Pfn1B'P). [*263-51.*206-25.*211P726] D. (ax). x Ec(I'P. Pfn'B'P u t'B'P- P'x: D F. Prop *263-53. F:Pc12infin-coi..Nr'P.>w Dem. F. *253-13. *263-52.:) F:: Hp.:). P ~(Pfn'B'P v 1'B'P) eDIP,. [*263-5]:). a!w DIPs. [*255-17.*263-18] D. Nr'P > co: D F. Prop The above proposition shows that co is the smallest of infinite ordinals. The samne fact is otherwise expressed by the following proposition. *263-54. F:a~E NO infin -tfco.).a c>o, [*263053] *263-55.FPo,)Peo-. Pw — Dem. F.*253-511.*263-44.:)F:Hp.:).Psew-~i- (1) F.*252-372.*263-44.:)F: Hp.).s'Peo-~i- (2) F.(1).(2).) F. Prop

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154 SERIES [PART V The following propositions are lemmas for proving *o- 2r c (*263-63). *263-6. F:: PeSer.ax+y. M= P x(x,y):. RMS.= (HU). U E CMP. R = x 4, u. S = Y U. u*V. (HU, V). uPIv. B yU. S x 4, v Dem. I-. *166-111. D F:.Hp. uPv. R = wx4 u: Sw 4, v I v. v. S~,v: -RM (y 4, u). (y 4, u) MS. [*201-63.*204-55]). (RM1 S)(1 Similarly F:.Hp.uPv.R = y,u.S = y Iv.(RMS) (2) F:Hp. uPw. wPv. R = y 4, u. S = x 4, v. D. R M (x 4,w). (x 4, w) MS. [*201P63.*204'55] D. (BM1S) (3) F-. (1).(2). (3). Transp.*1 66-111.) F-:. Hp. RM1S. -: (5ju). B =w4, u. S =y 4, u. uEU'P. v. (HU, V).uPiv. R =,y4U. S= x Iv (4) F. *166-111. DF: Hp.BR= x4,u. S= y4,u. M(x4, v).D. SM(x4Iv) (5) F. *166-111. D F:. Hp.- R= x 4, u. S=y 4, Iut. RM (y 4, v).): u =v *v. uPv: [*166-111] D: Y4,v=-S -v.SM(y4,V) (6) F: Hp. R= y4u. S = x4,v uP~v.BRM(y4, w).:).SM(y4, w) (7) FI- *166-111.)3 F:. Hp.- R =y 4, u.5S= x 4, v. uP~v. RM~x 4, w).): x4,W=S.v.SM~e4,w) (8) F. (5). (6). (7). (8).)3 F:. Hp: u e G'IP. B x 4,u. S =y 4,u v.uP~v.BR~y4,u. S = xv::).RM1S (9) F. (4). (9).) F. Prop *263-61. F:PESer.x~y.M-Px (x4y).:). U'M1==y,"G'P v x4,"U'1P1 [*263-6] *263-62. F:Pcco-xty.:).Px(w4,y)EW(Dem. F. *2163-61-43. D F: Hp.:). (1Px (xv 4,y)}]y4"',vc,C' 1*166-111] =WI'P x (x4, y)} (1) F.*251P55. DF:Hp.D.Px (x4,y),EfI (2) F.*166'14. DF:Hp.).Px~v4,y)c-t'A (3) F. *166'16. *263-22. DF: Hp.:). BCnv'{[P x(x4,) A (4) F. (1).(2).(3).(4).*263-44.) F:Hp.).P x (x4,y) e::)F. Prop *263-63. F. ' k2r - Dern. F. *263-62-17.)F:PREo. Qc2,).Nr'(P xQ) =w (1) F. *1 84'13. *263 17.)DF:P cw.Q c2,). Nr'I(P xQ) =co*k2, (2) F. (1). (2). -DF: a! w. a2! 2,. (o*2,= (3) F. *184-11. D F:co =A.). co*,72r== A (4)

SECTION E] PROGRESSIONS 155 SECTION2E] PRGESIN 155 F.(3). (4). (5).) F. Prop The following propositions are lemmas for proving *263-66. *263-64. F:P, Qe6Ser.xeC'P.zQ~w.M=PxQ.).(z4,x)M1(w4,x) Dem. F.*166-111.)I: Hp:).(z4,x)M (w4,x) (1) F.*166-111.)F:.llp.(z4,x)M(u4,y).):xPy.v.x=y.zQu: [*204-71]):xPy.v. x=y.u=w. v.x=y.wQy: [*166-111]:): (w w) M(u I y).V. (IVIx) = (u y) (2) F.(2). *20455. DF:Hp (2).).r { (u I y) M(w I x)} (3) F.(I).-(3). *201-63. DF. Prop *263-641. F:P,QeSer.z=B'Q.w=B3'Q.xP~y.M=Px Q.). Dern. (z 4wX) ml (W4,) F. *166-111 FI: Hp.:).(z4,x) M (w4 y)(1 F. *166-111. DF:. Hp. (z x) M(u4, v).):xPv: [*204-71]:v = Y.V. yPv (2) F. (2). *166-111. D F: Hp. (z 4, x) M(u 4, v). D: u 4, v = w 4, y. v. (w 4, y) M(u 4, v): [*204-55]): {u 4, v) M(W 4, y)} (3) F(1). (3). *201-63. DF. Prop *263-642. F:F, QESer.M=P xQ.).(C,'P x G'Q,) CG'MI [*263-64] *263-643. F:P,QESer.E!B,'Q.E!B,'Q.M=PxQ.).(B'Q)4,"U1'PCPIMI [*263-64] *263-65. F: Pcco. Qef16Un-tA.).P x Qe Dem. F.*251P55.DF:Hp.3.PxQef2 (1) F.*166-14.DF: Hp.:).PxQc-t`A (2) F. *263-642i643. *261-24.)D F: Hp. D. (CP xU('Q,) v(B'Q) 41"(PP1 CP(PI x QXI. [*263-49]).(G'P x W1Q) v (B'Q) 4,'WJI'P C G'(P x Q),. [*166-12-16]). G(P x Q) - B'(P x Q) C U'I(P x Q)1. [*93-101.*201-63]). U(P x Q) = CU'(P x 9), (3) F.*166-16.*263-22.) F:llp.). B'Cnv'(P x Q) =A (4) F. (1). (2). (3). (4). *263 44. D F. Prop *263-66. F:aENOfin-t'O,.:).w~ka=eo [*263-65] The proof proceeds as in *263-63.

*264. DERIVATIVES OF WELL-ORDERED SERIES. Summary of *264. The principal purpose of the present number is to show that every infinite well-ordered series is the sum of a series of progressions followed by a finite tag, which may be null. For this purpose, we proceed as follows: If x is any member of C'P, it must belong to the family, with respect to PI, of some member of C'P -(PPI, unless x=B'P and B'Po eCI'PP. Assuming that we have either E! B'P or B'P e (Pi, and assuming further that P is al infinite well-ordered series other than a progression, it follows that every member of C'P belongs to the family, with respect to PI, of some member of C'V'P, because, by *216-611, C'V'P = D'P1 - 'P in the circumstances contemplated (*264'15). Now P limited to any one family with respect to Pi is a progression, unless that family includes B'P; and if it includes B'P, it is finite. Hence our proposition follows. An important consequence of the above proposition is that every cardinal which is not inductive and is applicable to classes that can be well-ordered is a multiple of K0 (*264-48). For the purposes of this number we need a notation for the series of series each of which consists of the family of some member of C'V'P. We therefore put Ppr=P;(P)*;V'P Dft [*264]. Here "pr" is intended to suggest "progression." When P e 1 infin -o, Ppr is the series of progressions (possibly ending in a finite tag) whose sum is P (or Pt D'P, in one case). Before using this definition, some preliminary considerations are necessary. V'P is the series of limit-points of P, including B'P. In order that V'P may exist, there must be at least one limit-point besides B'P. Now the limit-points of a series are C'P- I'PP, i.e. the limit-points other than B'P are ('P- (P, (*216'21).

Hence when B'P exists and CI'P- ('P exists, V'P exists. Hence by *263-49, *264'13. F:. Pe 1. 3:! V'P. =. Pe f infin - w

SECTION E] DERIVATIVES OF WELL-ORDERED SERIES 157 I.e. a well-ordered series whose derivative exists is one which is infinite and not a progression. We have similarly *264-14. H: P e infin -.. C'V P= C'P - 'P and *264'12. F: P e. D. aPV'P = C'P - ('P1 We next proceed (*264'2 —261) to study the posterity of a term x with respect to Pi, i.e. the series P t (Pi)*'x. We show that if this series has a last term, it is finite (*264'21), and ends with B'P (*264-2), while if not, and if xeC'PP, i.e. if x has either an immediate successor or an immediate predecessor, the series is a progression (*264'22). Hence we have *264'23. F:. P e. x e C'V'P n C'P1.: E! maxp'(Pi)*'x. =. x = B'Cnv'V'P. E! B'P Moreover, if xc C'P1, the ancestry of x with respect to Pi must end with a member of the derivative of P, i.e. *264'233. F: P e f infin - o. x e C'P,.). minp'(Pi),x e C'V'P We thus arrive at the result that if P has a last term, so has V'P (*264'24), and if x is any member of the derivative except the last, the 4 --- series P t (P,)*'x is a progression (*264'25), while if x is the last term of the derivative, and the series P has a last term, then P t (P,),'x is finite (*264-252). Moreover the supposition that P ends with a member of the derivative is equivalent to the supposition that P ends with a term which has no immediate predecessor (*264'26). We now proceed (*264'3 —403) to consider the relation Ppr defined above. If we take any term y in a well-ordered series, there is some term x belonging to C'P- G'PP such that the family of y with respect to Pi is the posterity of x. This results from *264*233 above. Thus we may divide the field of P into mutually exclusive stretches, each of which is the posterity of some member of C'P- I'P, with respect to Pi. The series of series thus obtained is Ppr. There is an exceptional case, when the series ends in a term having no immediate predecessor, for then the posterity of this term with respect to Pi is null, and therefore Ppr omits this term. Otherwise, we shall have 'Ppr= P; i.e. we have *264'39.: P e I infin - o. (B'P e C'V'P). D. X'Ppr = P *264-391. F: P e f. B'P e C''P. D. S)Pp, = P D'P Moreover we have *264-36. F: P e f.). Ppr smor V'P. Ppr e Rela excl

158 SERIES [PART V From what was proved earlier we know that, assuming P e f, we have D'Ppr C C (*264-401); if P has no last term, C'Ppr C o; if P is infinite and has a last term, B'Ppr is finite, and if the last term of P belongs to C'V'P, B'Ppr= A. Hence, using *251'63, which assures us that, in virtue of *264'36 above, if C'Ppr C w, 'P'pr is a multiple of w, we find (*264-44) that every well-ordered series has an ordinal number of the form (ax o) -/3, where a and / may be any ordinals, including 0r and i (putting i X a = a to avoid exceptional cases). The above account omits the exceptional cases, which require special treatment and render the proof long; but in the end the above simple result is obtained.

Since a multiple of o0 is not increased by the addition of an inductive cardinal, it follows (*264'44) that the cardinal number of the field of an infinite well-ordered series is always a multiple of No (*264'47). Hence if all classes can be well-ordered, all cardinals which are not inductive are multiples of K. In virtue of Zermelo's theorem, the same result follows if the multiplicative axiom is true. *264'01. Ppr = P C;(P;V'P Dft [*264] *264'11. F:. P e gI. D: a! sgm'P. -. P e f2infin Dem. F. 263-51. D F: P e I infin. 3.! sgm'P (1) F. *212-152. *211'41. 3: Pe f. [t! sgm'P.. X! sect'P - t'A - Cmaxp. [*261'28.Transp] 2. P e f infin (2) F. (1).(2). D F-. Prop *264-12. F: P e f. D. ('V'P = I'P - ('P1 Dem. F. *216-61. F: Hp. [! P.. (D'V'P = a[P - a(P (1).*216'612.*33-241. 3 h:P =.. 'V'P = A. ('P- ('P =A (2) F. (1). (2). 2) F. Prop *264-13. F:. P e.:! V'P. -. P e infin - Dem. F.*264-12.:):.Hp. D: [! V'P. -.! (['P- ('P,. [*263-49] =. P e f infin - o: D. Prop *264-14. F: Peflinfin-o.. CG'V'P=C-'P-([P1 [*264-13.*216-611] *264'15. F:. P e infin - w: E! B'P. v. B'Pe ('P1::. C'V'P=B'P Dem.. *264-14. *93-103.: Hp. E! B'P.:.C'V'P=C'P-(IP,. C'P=D'P. [*93-101.*250-21]:. C'V'P= B'P (1)

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SECTION E] SECTIN E] DERIVATIVES OF WELL-ORDERED SERIES15 159 F*93-101.) I-BP e (PPD.)C'IP - GPP1CD'P (2) F. (2). *264-14. D F: Hp. BIP e PP.P:). C'V'IP =D'PT-G'P1 [*93-101.*250-21] = B',P, (3) I-.(1).3).) F-. Prop 4- 4 — 1 *264-2. I-: P E 12. E! max p'(Pi)*'x. D. maxp'(P,)*'wx = B'P Deni. - 4 —4- 4- - F. *206-4246.) F: Hp.:). seqp'(P1)* 'X = Pi (maxp'(Pi)*'x. [*90-16] D. seqp'(P1)*'x C (Pi)*'w. [*206-2]).seqp'(P1)*'x = A. [*250-126] D. maxp'(Pi)*'x = B'P: D F. Prop *264-21. F:PcEfl. E!maxp'(P,)*'x.:). P ~ (P,)'x e f~ fin. P (x i —i B'P) E Cls induct Dem. 4 — ~~~4 — F. *200-35.:) F: Hp.(Pi)*'x = OX. D. P ~(PI).*'w 1 4 — 4 — F.*260-27. D F: Hp. (P,)*'x 4 t'x.:). xP~fn maxp'(P,)*'w. [*260'11] D. P Ix~-maxp'(Pi)*'xi E Cls induct. (2) [*205-2]). 'P ~ (P1)*'x E Cis induct (3) F. (1). (2). (3). *264'2. D F. Prop 4 — 4 — *264-22. F:Pl.-.Ettx'P),.e~,:.~P),C Dem. F. *260-3234-27.) F: Hp. D. lp~(,*Xl= f(I*tl1 p1. (1) [*122-52]). {P (P1)*'xji E Prog (2) F. (2). (3). *263-1) DF. Prop *264-221. F:Pe&f1.x(V'P)y.:).P(x-y>-.iEClsinduct Dqm-~~~~~~~~F.*207-34.*216-6.:)F:Hp.:).xP~y y-ltp'lP'y. [*207-25] D. Xpl2y. y = 1tp'(P'x A P'y). [*207-13]. P2y.' E! maxP'(P'x AP',y). [*261P26]).P'x n P'y eje Cls induct: D F. Prop *264-222. F:P el. P'x eCisinduct.) x r. e D'V'P [*264-221. Transp]

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160 SERIES [PART V *264-223. F:Peil P (-,y),E Cls induct.).a!UV'PAnP (x -ly) Dem. F -.*261P3.:) I- Hp.:). (Ha). a C P (x - y)* a a. — E! mnaxp'a. [*206-213] D. (aa). aC P(x -y).H! a. ltp'ae P(x — i y). [*206-181] a! D'ltp d'(IP ^P (x -— ' y). [*216-602])a!P1V'P P (x — y):)IF. Prop *264-224. I-:PE&2.x=B'Cnv'V'P.E!B'P.). P'xEClsinduct Dem. F. *264-223. Transp.) IF: Hp.). P (x - B'P) E Cls induct:) F-. Prop *264-225. I-:. P e *.x eC'J?.: E! maxp'(Pi)*'x.-. (Pi)*'x eCis induct [*264-2122] *264-23.F H. PE flw EO'V'P riC'P1.) E! mnaxp'(Pi)*'x. *x B'CnvV'VP. E! B'P

Demn. F. *264-2.)IF Hp.E!maxp'(P1)*'x.)D.E!B'P (1) F.*2 6421P2 22.)iF IIp (I).).-x'-. e D'V'P. [*93'103] D.x = B'Cnv'V'P (2) F..*264-224.) F- Hp.x = BCn v'IVIP. E! B'P)P'x eCls inducet. [*120-481'251]).(P1)*'x e Cls induct. [*90-12.Hp. *261-26] D. E! mnaxp'(Pi)*'x (3) F-.(1).(2). (3). IF. Prop *264-231. I-: P Ef2.xE 'V~P -C'P,.).x =B'Cnxv'V'P=-B'P Dem. F-. *250-21.)FI-:Hlp.).xE D'P. [*216-6] D) x.W e D'V'P. [*93-103]).x = B'Cnv'V,1'P (2) F-.(1).(2).)IF. Prop *264-232. F:. P ei2.-x eG'V'"P.) (Pi)*'x eCis induct. -. x =B'Cnv'V'P. E! BIP This proposition differs from *264k23 by not assuming that x e C'P,. If B'P has no immediate predecessor, B'P EC'V'IP -C'P1, so that B'P satisfies the hypothesis of *264-232, but not that of *264-23.

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SECTION E] SECTIN E] DERIVATIVES OF WELL-ORDERED SERIES16 161 Dem. F. *90-13.:)F:Hp.(P,)*'x=A.:).x —,,CGP,. [*264-231]).x = B'Cnv'V'P. E! B'P (1) F. *120-212.) I-: Hp (1).). (P,)*'xwe Cis induct (2) F.- 264-225.) -. H1p. a! (P,) *'x.): (P1)*'x e Cis induct. _= E' maxp'(Pi)*'x. [*264-23] =Ex=B'Cnv'V'IP.E!B'IP (3) F. (1).(2). (3).) F. Prop *264-233. F: P E il infin - w xE cP GP. D. minp'6(Pi) *'x E c'v'rp Dern. F. *250-121.): IHp.:). E! minp'(P1)*'x (1) F. *90-172.)F: Hp. y (Pi)*x. zPiy.)zE6(Pi)*'X %P'y. [*205-14]).y [mirlp'(Pi)*'x (2) F-. (2). Transp.) FI Hp. y = miipl'(Pi)*I'x.) -. y E cL"P1. [*264-14])YeG'~V'P (3) F()(3).) F. Prop *264-24. F: P ei7~infln.E! B'P.:).E!B'Cnv'V'P Dern. F. *264-12.)D F: Hp. B'P,E C'P.) D.B'PEG'6V'"P. [*216-6] D.BITP=B'Cnv'11V'P (1) F. *264-233.*263-22.)F: Hp. B'Pe C6P,.).minp'(P1)*'B'PECG'V'P (2) F. *205-55. D F: Hp (2). x = tninp'(P,) *'B'P.).B'P = maxp'(P1)*'x. [*264-23.(2)])x = BCnv'1V'"P (3) F.(I). (3).)DF. Prop *264-25. F:Pe6f1.xeD'V'P.).P~(PI)*'XEw Dern. F.*264-232.*250-21.:)F:Hp.).(Pi) *'xE Cis induct.~e ED'P,. [*264-225]). E! rnaxp'(P1)*'x. x E D'P1. [*264-22]).P ~ (P,)*"x e: F. Prop *264-251. F ef.,E I.x ~,p.P~(,*X6c Dem. F. *250-21. F: Hp.. xe6 D'P1. [*264-23.Hp]). E! maxr'(Pi)*'x. x E D'P1 [*264-22] D. P ~(P1)*'x e::) F. Prop R.& W. III. 1

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162 SERIES [PART V *264-252. F:PEf~.E!B'P.x=B'Cnv'"V'P.:).P~(P%*'xcflfin Dem. F. *264-23.) F: Hp. x E C,'P,.). E! maxp'(Pi)*'lx. [*264-21])P ~(P,) *'xwe&2 fin (1) F. (1). (2). DF. Prop *264-26. F:.PEf~.D:B'PeC'"V'P. —.E!B'P.B'P,. sd7I'PI Dern. F.*14k21.:)F:B'lPcG'V'P.:).E!.B'P (1) F *264-12.:)F: Hp. B'P cC'V'P - D.)B'P,EUIP, (2) F.*264-12.)F:Hp.B'P —eG"P1.).B'1%EC'V'7P (3) F. (1).(2). (3).) DF. Prop *264-261. F:. P cf2.:~(B'P fG'V'P). n. C'P =GcP1 Dem. F. *264-26.:)F:: Hp) (B'PEGIV'P). -sE! B'P.v. B'P (TIP1% [*202-52] nBITPCU('P,: [*250-21] G'P C G'P1: [*121P322] C'P = C'],1:: D F. Prop *264-3. F: QPrR. (3).xVPY. Q=P ~ (P,) *'x. R =P ~ (P1)*'y [(*264-01)] *264-31. F:. PecSer.): QPprR (ajx, Y). x,.yc C'P - (I'P1 xPy. Q = P ~ (Pi)*'wx. R1=P (1)' [*207'35. *264-3. *216-6] *264'32. F G'P~pr = P ~t"(P1)*'W'(V'P [*150-22. (*264-01)] *264'321. F::PeSer. x eG'V'. P)" Dem. F. *216-611.:)F: Hp.D. x cC'P-U'1P1 1 F. *121-305.:) F: Hp. x eDIP,. D. aj!

(P,)*'x - t'x. [*90-12].(P1)*'Xrsel6 (63) F. (1) (2). (3). DF. Prop *264-33. F: P ie Ser. D. G'P'"CPpr = (P1)*"C'cV'P [*264-321. *202,55. *264-32]

SECTION E] SECTIN E] DERIVATIVES OF WELL-ORDERED SERIES16 163 4- 4 — *264-34. F:PEfl.x,yeG'P.P~(Pl)*'x=P~(Pl).*'y.).x=y Dem. F. *264-321. *202-55.) F:Hp.. (P1)*'x = (P1)Vy (1) F. (1). *90-12.)F Hp.wxEG'PI.).x(PI)* y. y(P,)*x. [*260-22.*91-541]). = y (2) F. *250-21.)F: Hp. x, —E G'P,.).cc=BP (3) F. (1). *90-12 14.) F:Hp. x c C'P,.)y'-.eG'P1 [*250-21]).=B'P (4) F. (2). (5).:)F. Prop *264-341. F:PESer.x,YEC'lV'P.x(P1)*y.:).x=y Dem. [*91V504]). { x (P1)P0y}. [*91-54] D. x= y: DF.Prop 4 — 4 — *264-35. F:PcSer.x,yeC'V'P.a!(Pi)*'xrA(Pi) *'y.:).a=y Dern. F. *960302.:) F:. Hp) D: x (P,)*y. v.- y (P1)*x: [*264-341]:): x=y:.:)F.Prop *264-36. F: Pc El tI). Pprsmor VIP. Ppr E Re 1 exci [*264k34-35] The following propositions lead up to *264'39-391. *264-37. F:P e i2infin - co. D. 4"Ppr=Pfn Dem. F. *264-32. D F:. Hp.:)::X(C'Ppr) Y.(3a). aE C'IV'P. x, y (P,)*la. xPy. [*260-32-27].(2ta). a c C' V'P. x, y E(Pl)*'Ia. XPfny.[*264-233-35].(2ga). a = rnin'(1P0)*"X = rninP'(Pl)*'Y. XPfny. [*13-195].rinp'(Pl)*'X = Irnin'P1)*'Y. XPfnY (1 F. *260-27.:) F: Hp. xPfny.) (Pi)11"x C (P1)*"y. [*205-5]).minP'(P1)*'X = minp'(P1)*'IY (2) F. (1). (2). D F:. Hp.): x (A'C'Ppr) y. xPffly:.)D F. Prop *264-371. F: P E Ser. a (V'P) b. D. (P1)*'a C P'~b Dern. F.*216-6. F:Hp.).aEP'b (1) F. *204-71.)F:Hp.xPb.xP~y.e,(~yPb).:). y =b. [*33-14]). b (1PP (2) F. (2). Transp. *216611. DF:. Hp.): xPb. xP~y.D. yPb (3) F. (1). (3). *90-112.)D F:. Hp.:): a(P1) *x.:). xPb:.) F. Prop 11-2

164 SERIES [PART V *264-372. F: PeSer. D F;Ppr C P -LPfn Dem. F-. *264'3-321. *202-55.) F-:. Hp. Dx(F;Pp,) y. -.(ga, b). a(V'P) b. x C(P,)*'a.yc (P)*'b. 1 [*264-371] D. xPy _ _ (2) F-. *264-35. DIF Hp. a (VIP) b. XEC (P1)*'a. y e (P1)*'b. D. e (Pi)*'a. [*260-27]). (XPfn Y) (3) F. (1).(2).(3). F: Hp.)FiPpr C P -~-Pfn: D F-. P rop *264-373. F-:Pefl.r-...,(B'PeC'V'P).).P-!-PfC-F;Ppr Dem. F-. *264-261'233. *263-49.)D F-: HP. x(P —Pf11).y.).minip'(Pi)*w, miiip'(Pi)*'y EC',V'P() F-. *96-301.)F-:. Hp. minp'(P1)*'x = rinp'(P1)*'y.) x (P1)*y.VYP) [*260-27] __:X=y-V-XPf,,y-V-yPfnlX (2) F-. (2). Transp. D F- Hp (1). D. minp'(Pi)*'X t_'lli~p'(P,')*'y (3) F-. (1). *264-371.)D F- Hp. minp'(P1)*'ly P minP'(P1)*'wX. D. yPw (4) F-. (4). Transp.) F-Hp (1).). -, minp'(P,)*y P Minp'(P,)*'XI (5) F-. (3). (5).)D F-: Hp (1). D. iminp'(P1)*'x P minp'(P%)*'y (6) F-. (1).(6).)F-:Hp (1).). (ga,b).a (V'P)b. x E (P,)*'a.y ~ (P,)*'b. [*264-3-321.*202-55)]). X (F;Ppr) y:) F-. Prop *264-38. F-: PE&2.-(B,'PeC'V'P).:). FPpr=P- Pf,, [*264-372-373] *264-381. F-:PE2I.B'PeC'V'P.:). FPpr=P~D'P-'.-Pfn, Dem. [*264-26.*42-2] D. B'P C GF; Ppr. [*264-372]:) F; Ppr CP ~D'P Z-Pfn (1) F-.*250-21l.)F-:Hp. x(P~ D'P?-!Pfl)Y.).X, YEC('P1. [*264~233. *263-49]:). minp'(Pi)*'x,' minp'(I%)*' Ic G'V'P (2) Thence as in the proof of *264-373, F-. (1).(3).) F-. Prop

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SECTION E] DERIVATIVES OF WELL-ORDERED SERIES 165i *264-39. F: P e f infin - o.(B'P e C'V'P).).'Ppr= P [*264-3738.- *26012. *162-1] *264-391. F: Pdf2. B'P e C'V'P. D. Y'Pr, = PC D'P Derm. F. *264-13. F: Hp.)D. PE &il infin - w (1) F. *260-27.) F Hp.. Pf = Pf, n CGP1 [*264-26] = Pfn C D'P (2) F.(l).(2). *264-37. *26O-12.:)F: Hp.)..9U'Pr = Pfn. Pfn C P D'P (3) F.(3). *264<381.:) F. Prop *264-4. F:Pef~ *(-,)E!B,'P.).C'PprCo [*264-251P32] *264-401. F: P c fl. ~. D'Pp C co Dem. F. * 51-5. *26434 34.:) F: Hp - D D'Ppr = P c((Pj)*ccDc VP (1) F. (1). *264-25. DF. Prop *264-402. F: P e f2 infin. E! B'P. D. BI'PP ef~2 fin Dent. F. *264-24.) F: Hp..E! B'Cnv'V'P. [*151P5.*264]43]). B'PPpr=P (P, '"B'Cn vIV I'P. [*264252]). BCPp elfinfln:)F.Prop *264-403. F: P fl2. B'P e C'V'P.)IP B'PprA Dem. F. *264 26 231.) F: Hp.).B'P e C'P,. B'P = BCnvIV'P. 4 --- [*90-14]. (P,)*'B'Cnv'V'P = A. [*1515.*26434]) B'Pr =-A:)F.Prop The following propositions deal with the various different cases that arise. Their net result is expressed in *264-44. *264-41. F:Pe infin-.o.,E!B'P.).Nr'P=Nr'V'P,,< o Dem. F.*264-36A4.)F:Hp.). PPpReleexclnNr'V11P.C"Ppr Co. [*251i63]) - 'Ppr e Nr'V'P j'oW. [*264P39] D. Pe Nr'V'P i< w: D F. Prop

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166 SERIES [PART V *264-42. F:Pef1.B'P,-eC'V'P.V'Pe2,.:).Nr'P=o)-i-Nr'B'Pp. Dem. F. *264-36.) F: Hp.).-Ppr =(B'Ppr)4, (B'Ppr> [*1 62-3.*264-39-13]:).P =BPprt-B'Ppr. [*264-36-401])Nr'"P = w4B'Ppr DF. Pr op *264-421. I-:PEfI. B'PeC'V'P.V'Pe2..).Nr'P=o-~iI Dem. F. *264'36.:) Hip.)D. Ppr=(B'Ppr) I~ (B'Ppr). [*162-3.*264-391-13] D P~ D'P =B'Pprt-B'Ppr [*264-403.*160-21] = B'Ppr. [*264-401] D. P D'PRT~ [*204-461]:).Peo-~i-:DF.Prop *264-422. F:Pef~infin-co. B'P-,EC'V'P.V'Pce2,.:). Nr'cP {Nr'(\V'P)~ (D'V'P) * w} -INr'B'Ppr Dem. F.*264-36. *204-272.:)F: Hp.:). D'Ppr " 1. [*204'461.*264-24-36]).Ppr =Ppr ~ D'rPpr 1+ B'(Ppr [*162-43.*264-39]:).P= I'(Ppr~ D'Ppr) 4-B'Ppr (1) F.*264-36-401. *251-63.)D F Hp.). Nr'Y.'(Ppr~ D'Ppr) =Nr'(V'P) ~ (DI'V'P)*a c (2) F. (1). (2). *264-36.:)F. Prop *264-423. F:Pefk2.B'P cC' V'P "7P,E 2,. Nr'P = [Nr'(V'P) ~ (DI'V'P)k co] iDern. As in *264-422, F:HP. Ppr =Plpr ~ D'"Ppr -F> B'fPpr. [*I162-43.*264-391]:) P~ DIP Y-'(Ppr~ D'Ppr) t-B'Ppr [*264-403] -'Pr~ D'(Ppr) F. *264-36-401. *251-63. F-: Hp.). Nr'S'(Ppr ~ D"Ppr) =Nr,'(V'P) ~ (DI'V'PT) (co F.*204-461.) F: Hp.).Nr'P =Nr'(P ~D'P)F.(1).(2). (3).) F.Prop (1) (2) (3)

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SECTION E] DERIVATIVES OF WELL-ORDERED SERIES 167 *264-429. i;(a = a Df This definition is merely intended to enable us to include 1 with ordinals in general formulae. *264-44. F:P'f ~.).(gl).aENO v u'ti.3eNOfinvt '.Nr'P (aw) o)-i Dem. F.*160-22.*166-13.)F:Pef2fin.).Nr'P=(0r,;w)i-Nr'P (1) F. *160-21. D F: P = w).. Nr'P = (i k o)i- 0, (2) F. *264-41. *160-21. F~: P e f2 infin - E! u B'PP. D. (aa)

http://quod.lib.umich.edu/cgi/t/text/text-idx?c...stmath;rgn=main;view=text;idno=AAT3201.0003.001 (85 of 246) [5/26/2008 7:31:50 PM]

r). ae NO. Nr'P = (a;< o)- O, (3) F.266442-402. D F: P Ef.B`Pc-e C'V'P.V'PE2r.). (Jj) 3 e NO fin.Nr'P=(i *(wo) i-/ (4) F.*264-421. D)F: P ef~2.B'P e C6V'P.V'P e 2, r..Nr'P wiX)+I (5) F. *264-422-402.:) F: Pe f2 infin -o,. B'P e CV'P.P'P e 2r (aa,fl). a e NO. e NO fin. Nr'P - (aX< co) (6) F. *264-423. D F: P c2. B'P e C'V1'P. V'P, e 2r.)D (act).a eNO. NrIP = (a;O co) 4 -F. (1). (2). (3). (4). (5). (6). (7).:) F. Prop The following propositions apply the above results to the cardinal number of the field of a well-ordered series. *264-45. F:Pef2.V'Pe2r..).Nc'C'P = No Dem. F. *264424022. *18071. *1527.) Hp:Np. BIP N CcVIPP. D. egu.~ NC induct. NcICIP = C"w +,,,a [*263-101.*123-41]. Nc'C'P= K, (1) F. *264-421. *1 81-62. F: HP. B'P e C'V'P). Nc'C'P = C"o +a 1 [*263-101.*1234] = No (2) F. (1).(2).)F1. Prop *264-451. F:Psflinfin- co. E!B'P.).Nc'C'P= Nc'C'V'PXc o Dern. F. *264-41. *184-5.:) F: Hp.). NcWC'P = NcW 'V'P x0, C"eO [*26C:3.101] - Nc'C'V'P x No: D F. Prop *264-452. F: P e Qfinfin - co. V 'P e 2r.B'P 6 C'V 'P.). Nc'C'P = Nc'D' VP x0 No Dem. F. *264422. *1845. *180-71.) F: Hp.:. (ajj). pE NC induct. NcW'P - (Nc'D'V'P x, o) t, (1)

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 F.*1 23-43. *117,62. DF:Hp. jeNC induct.).p~< Nc'D"V'P x0 No. [*117-561]:). (Nc'D'cV'P x0, K0) +,/ < (Nc',D'V'P x,0 KO) ~c0 (Nc'D'V'P x0, N.) [*1 23 421.*113,43] < Nc'D'IV'P x. No (2) F. (1). (2). *117-625.:) F: Hp.) NcCP'P= Nc'D'V'P x0KN,::) F. Prop *264-453. F: P e fl infin - co. E! BP. V'"P,,e 2r.). NcIC'P = Nc'D'V'P x. No Dem. As inD*264-452, F.*264-423.)F:Hp. B'Pe6C'V'P.).Nc'C'P=Nc'DI'V'Px0K0 (1) F. (1). *264-452.:)F. Prop *264-46. F:Pc Q infir - c.). Nc'C'P=-Nc'C'V'P x0 K Dem. F. *123-421.*264-45.) F:Hp. V,'P E2,.). Nc'C'P= Nc'G'V'P xcKo (1) F. *264-453.) F: Hp. E! B'P. VP c 2r. Nc'C'V'IP = IL ~+, 1 Nc'C'P =Au xc, [*123-421.*113-43] (txc MK) +c, (xc,,K) (2) F.*1 17 -57V6. F: Hp), cN,<,(t+)x,(+)x o,(Lx, c(i O (3) F.(2). (3).) F: Hp..N'(J'P= (Att+c1) x0K0N [Hp] = NcPCV'PT x~ K0M (4) F. (1). (4). *264-451..F. Prop *264-47. F: Peffinfin.:). (Hj).,4cNC-tffO.Nc'C'P=At x0K0 [*264-46] *264-48. F: a 6C I- CI-isinduct.). Nc'la cD'(x, K0) [*264-47]

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*265. THE SERIES OF ALEPHS. Summary of *265. In the present number, we shall confine ourselves to the most elementary properties of the ordinals and cardinals considered. The most important propositions to be proved are the existence-theorems. These all depend upon the axiom of infinity; moreover, as the numbers concerned grow greater, the existence-theorems require continually higher types. In virtue of the definition in *262, (Ko)r is the class of well-ordered series whose fields have Ko terms. This is not an ordinal number, but the logical sum of a certain class of ordinal numbers, namely of Nr"(Ko)r. w0 is the smallest ordinal whose field has more than K0 terms. We do not, however, take this as the definition of o,: we define co as the class of relations P such that the relations less than P (in the sense of *254) are those well-ordered series which are finite or

University of Michigan Historical Math Collection: Principia mathematica, by Alfred North Whitehead ... and Bertrand Russell.

have N, terms in their fields, i.e. w, = P {less'P = (No), u fn fin} Df. By *254'401 it follows immediately that if Pe (o, P is a well-ordered series and Wo is its ordinal number (*265*11). Hence co, is an ordinal number (*265'12), though we need the axiom of infinity to show that 6o, exists. Assuming the axiom of infinity, the existence-theorem for wc is derived from the series of ordinals which are finite or belong to series of Ko terms. For notational convenience, we temporarily define this series as N; thus N=(<E) {NO fin u Nr"(Ko)} Dft [*265]. It is also convenient temporarily to write M for " < ": thus M= < Dft [*265]. It is easy to prove that if K0 exists, N is an w, (*265'25). Hence we obtain the existence-theorem for oI in either of the forms: *265-27. F:! 0 n t'a.).a! n tot"'t'a *265-28.: Infin ax (x). D). [! w r tI't33'x It is easy to prove that wI is greater than the ordinal number of any series of Ko terms (*265'3), and that if wc exists, M = NO fin v Nr"(Ko)r (*265-35), i.e. the ordinals less than o, are those that apply to series of Ko terms or of a finite number of terms.

 We define KN as C"weo, i.e. as the class of those classes which can be arranged in a series whose ordinal number is o01. It follows from *152'71 that KM so defined is a cardinal number (*265'33), and that if Ko exists, Ml > Ko (*265-34). In a precisely analogous fashion we can put A -e 2, = P {less'P = (R), (Mo)r u f fin} Df, K2 = C"o2 Df. Theorems similar to those mentioned above can be proved for oW and K2 by similar methods. We can proceed to co, and K,, where v is any ordinal number. But our methods of proving existence-theorems fail if v is not finite, since at each stage the existence-theorem is proved in a higher type and we know of no meaning that can be assigned to types whose order is not finite. It is easy to prove that the sum of two ordinals which are less than Wo is less than w,. Much of the accepted theory of (K0)r and wco depends upon the proposition that the limit of any progression of ordinals less than wo is less than o,, so that in the series N, every progression has a limit within the series. This proposition-or at any rate the current proof of it-depends upon the multiplicative axiom. The proof, in outline, is as follows: It is easy to prove that an ordinal which has Ko predecessors must be a member of Nr"(Ko)r, i.e. must be, in Cantor's language, an ordinal of the second class. Now consider any progression P contained in N1, i.e. consider a series a,, a2,... a,,... of increasing ordinals of the second class. The interval between any two consecutive terms of this series is either finite or has Ko terms. Hence N"C'P, i.e. the class of ordinals preceding the limit of our series, is the sum of Ko classes each of which is finite or has Ko terms. It is then argued that, because K0 x K0 = K0, the whole class N"C'P must consist of Ko terms. This conclusion, however, except in special cases, requires the multiplicative axiom, since it depends upon *113'32, i.e. F:. Mult ax.):,, v e NC. K e v n C1 excl'pu.). s' e el x, v. It follows that, unless for those who regard the multiplicative axiom as certain, it cannot be regarded as proved that o,1 is not the limit of a progression of smaller ordinals. With this, much of the recognized theory of ordinals of the second class becomes doubtful. For example, Cantor proceeds to define a host of ordinals of the second class as the limits of given series of such ordinals. It is probable that, in regard to all the

http://quod.lib.umich.edu/cgi/t/text/text-idx?c...stmath;rgn=main;view=text;idno=AAT3201.0003.001 (87 of 246) [5/26/2008 7:31:50 PM]

ordinals which he has defined in this way, a proof that they belong to the second class can be found, by actually arranging the finite integers in a series of the specified type. But the mere fact that they are limits of progressions of numbers of

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SECTION E] THE SERIES OF ALEPHS 171 the second class does not, of itself, suffice to prove that they are of the second class. As another example we may mention the very interesting work of Hausdorff*, much of which is based upon the proposition that a term which is the limit of an wo chosen out of a given series cannot be the limit of an w chosen out of the same series. This proposition is a consequence of the proposition that w0 is not the limit of a progression of smaller ordinals, and must therefore be regarded as doubtful. Hausdorff constructs by means of it many remarkable series, for example, compact series in which no progression or regression has a limit. The existence of such series appears, however, to be open to question, unless the multiplicative axiom is assumed. It is not improbable that a proof, independent of the multiplicative axiom, can be found for the proposition that w1 is not the limit of a progression; but until such a proof is forthcoming, the proposition cannot be regarded as certain. *265'01. w1 = P Iless'P = (o)r v fI fin} Df *265 02. MI = C"w, Df *265'03. 2 = P {less'P = (I), v (o)r, n fin} Df *265'04. K = C"o2 Df etc. *265 05. M=< Dft [*265] This definition is revived from *256. *265-06. N= M C {NO fin v Nr"(No)r} Dft [*265] The existence-theorem for wo is derived from N, since, if No exists, Ne Cl. *265-1. F:. P e ao. =: Q less P. -Q. Q e f. C'Q e Cls induct v Ko [(*265-01)] *265-11.: Pe alo. D. o = Nr'P.Pe n Dem. F. 2265'1. 1 F: Hp. D. A less P. [*254'1] D. P e (1) -. *254401. (1). (265-01).:): Hp. Q e 1. D. Q smor P (2). *254'401. (1). (*265-01).): Hp. Q smor P. D. less'Q = (o)r v f fin. [()265-01)] D. eo, (3) F.(1). (2). (3).:) F. Prop * Untersuchungen fiber Ordnungstypen. Berichte der mathematisch-physischen Klasse der Koniglich Sachsischen Gesellschaft der Wissenschaften zu Leipzig, Feb. 1906 and Feb. 1907.

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172 SERIES [PART V *265-12. F.weo, NO [*265-11.*256-54] *265-13. F:ae NO infin. 3.M ~M'ac a Dem. F.- *256 202. 3 F P Ec fl infin.). Nr'M ~ (M'Nr'P) =Nr'(P (U'P) [*262-112] = Nr'P (1) F.(1). *262-11.)F. Prop *265-2. F. PN= NO fifl-t'O v Nr"C(N)r=NAT~r [*255051] *265-21. F:a!N,.aeN~fin vNr"1(No)r.:). M ~M'a less N. aM (Nr,'N). a Cless'N Dem. F. *253-13. *265-2. D F: Hp. a E NO fin v NrI"(N0X.). M M'a e D%. [*254-182] D.M M'a less N (1) F. (1).- *265-13.) F: Hp. a E Nr""(No),. D. aM (Nr'N) (2) F. (2).- *263-31101. D F: lip. ac NO fin.)D. aMw. wM(Nr'N). [*256-1] D. aM(Nr'N) (3) [*255-17] D.a Cless'N (15) F.(1).(4). (5).) F. Prop *265-22. F:2!No.D). f~fin v (No)C less'N [*26.521] *265-23. F:PE D'N,.).(g [a).aeNO fin vNrc"(No)r.P=M~M'a. Nr'P= a [*265-2. *253-13. *265-13. *262-7. *120-429] *265-24. F: PcD'N,.).Pef~ fin v (N), [*265-23] *265-25. F:ag! o.:).

Nco, Dem. F.*254-41-12.)F:PlessN.:).(21Q).QeD'N,.PsmnorQ. [*265-24.*261-18. *151-18] D. Pe f2 finl v (No), (1) F.(1).*265-22.) F: Hp.).1ess',N=~flfin V(NO)r* [*265-1] D. New1,o:)D F.- Prop *265-26. F:a eNO.).Nor;(less ~ (If"C1') e o.Nor; (less ~C"Cl'a)=-N Dem. F. *254-431.*1 50-37.) F. Nor;(less C G"C1'a) - (Nor~less) ~ Nor"(f42 A C"C1'a) (1) F. *123-16. D F: a c o.:). N)r,"(f2 n Cf"CI'a) C NO fin v Nor"c(No)r (2)

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SECTION E] THE SERIES OF ALEPHS 1733 F. *123-14. *262-18&21.:)F: ad'N0,./e NC induct-L'1)- [/I CLrGC1'a: [*262-25] D F: acEN,. vE NO fin. D.ag! v n 0C"C1'1a. [*152-45]). v Ngr"C"CL'a (3) F.*1527. D F: PE (No),. ae No. D. a eCll"Nor'P. [*60,34]. Nr'P e Njr"G"Cl'a (4) -. (3). (4). D F: a e N. D. NO fin v Nr"(K,), C NXr"'("CL'a nr 11) (5) F.(2).(5). D F: aEK0. D. NO fin v Nr"(,),r=N,r"1 (GCl'(n f2) (6) F.(1). (6). (*255501.*265-0506). F: aD.). N,r;(less C G"Clla)= N. [*26525] D. Nr;(less C G"Cl'a) w: D F. Prop *265-27. F:a!NK0t'ca.).[!cow1 t1't.'a Dern. F.*64-55. F F:/e t'a. G'P Cf. D. PetOa (1) ():)F/E t'cx. D. C"CIfl C t0'a. [*1 55-12.*63-5] D. Nr""CCLC1'3 C t't00'a. [*6457]). Nor;(less C C"6C1'3) E t"'tXa (2) F.(2). *265-26. DF. Prop *265-28. F: Infin ax (x).. D - co, n t"'t3"x Demn. F. *123-37. F: Hp. a! N" n t't3'X. [*265-27] D g! WI n t)l!toolf A [*64-312] D! wI A t11'tm'x: D F. Prop Propositions concerning N, and w,, and generally N, and w,, where v is an inductive cardinal, are proved precisely as the above propositions are proved. There is not, however, so far as we know, any proof of the existence of Alephs and Omegas with infinite suffixes, owing to the fact that the type increases with each successive existence-theorem, and that infinite types appear to be meaningless. *265-3. F: a E Nr"(o)r.D. a < co) [*265-22-25] *265-31. F: ~IiNo N, >,, No Dem. F. *265-25. DF: Hp. D. CG'Ne (1) F. *265-2. D)F. NO fin - t'Or C C'N (2) F. *262-19-21. *123-27.)D F: Hp. D. NO fin - "r0 6 0 (3) F. (2). (3).) DPp: Hp. D. Nc'C'N > N (4)

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174 SERIES [PART V *265-32. F:ag! N0.:). Nt,+ N. NKn,= A Demn. F.*265302)] I-:E(.C'.).P E ml F. (1). *262-18. (*265-02).:)F. NK0AK1,= A.D J-.Prop *265-33. F. KXIE NC [*152-71. *265-12] *265-34. F:H~!NK.:). K1>N K[*265-31-3233. *255-74] *265-35. F!w,. M. Ao = NO fin v N~r"C(M)r Dem. F. *265-3. *2)6331.)F: Hp.) NO flu v N~r"C(M)r C M'W1 (1 F. *265-11. F: Pe E,. Nr'Q EM'w1,. D. Q less P. [*265-1]) Nr'Q e NO fin v Nr""(No)r (2) F.(1).(2).) F.Prop *265-351. F: P E co,..1! co,. Nr"ID'Pv = NO fin v NC(,) Dem. F. *256-11. *265-35.) F:!w.Nr" D'P, = NO fin v Nr"C(No)r. ~!wi M'N~r'P = M'w1. [*256-1.*204-34] E.P E co, F F. Prop *265-352. F:P E,.). N-r"D'P = M'w, [*265-35-351] *265-36. F:a 3cN"K~..a —/ Dem. F.*180-71.)F:Hp.:). G"l(a-i-3) = Cca ~, G" [*262-12] =N 0N [*123-421] =N [*262-12] D. a-i-/3e Nr"(Ko)r: D F. Prop *265-361. F.a, / E NO fin vNr"I(K,) r.). a -1-3E NO fin vrlCNr [Proof as in *265-36, using *120-45 and *123-41] *265-4. F: P e o,. aCC'P. P~caeCls induct v V)jp'Pcca Demn. F. *265-1.) F: Hp.:).

(P ~ P*"11a) less P. [*20 —51] *"a +CcP[*202-504] a! p'Pca: D F. Prop *265-401. F:PEco,.aCC'P.PI~aeCl'sinductvN0.:).a1!pC'PCa Dem. F. *205-131. D F: Hp. D a P*"C a = PC a v maxp'a. [*205-3.*12O-251.*123-4]:). P*"~a e Cis induct v K0,. [*265-4]). f! p'P~la: D F. Prop

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SECTION E] THE SERIES OF ALEPHS 175 *265-41. 1- P Eca~.).P"G'CP CN v Cisinduict. P*"G'CP C N, v Cis, induct Dem. F.*254-182.)F:lHp.):x eC'P.)(P ~P'~x) 1less P. [*265-1]).P'xE E v Cis induct (1) F.-(1). *1 20-251.*123-4.)DF:.IHp.):x eG'IP.)P*Ix e N.v Cls induct (2) F. (1). (2).) F. Prop *265-42. F:PecoD.(IJPCD'P Dern. F. *265-4-41.. F:Hp. xE (i'.). P p'P,(6/x. [*53-0131] D..x eDP: D F. Prop *265-43. F:lPewl.XEC'P.).P~Pf1i'xeco.E!tP'Pfn 'x Dern. F.*264-2.*265-42.) F:Hp.). - E! maxpgPffl'X. (1) [*264-22])-P ~ Pf,'w co (2) K.(2).*265'-41.*123-421.)F: Hp.:).P"Pfn'XsNO[*265-401] a! P'P"PIfnl'X F.(2). (3)) F.Prop *265-431. F:PCO1. QC-P-XEC'Q.Q'XCPfn1'X. J.3!PvP"C Dem. F. *265-43.) F: Hp.:). C'Q C P'tp'Pf.jx: D F. Prop *265-44. F:Peco,-xeC'P.).P~P*'wew, Dem. F.231.):p:.6P —*x~=R(?)x*yR PPx-) (1) F.*254-101.)F:Hp.xP*,y.:).Nr'P~P(xe-,y)~-<Nr'P~P'y. [*265-352]). Nr'IP ~ P (x i- Y) E M'w1 (2) F.*265-352.)DF:Hp.)D. Nr'IP P'X 6M"0,1 (3) F.- (3) - *265 -361-35 - F: Hp. a e Mfw. NrIP ~ P'x +a E M'w1o. [*265-35 1])(p)Nr'P Pixi-a =Nr'P ~P'y. [*253-47-11]).(dy). xP*,y.Nr'P ~P'x ~a = Nr'"P Py.

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176 SERIES [PART V [*204-45]:).(ay).xP~y. Nr'PP~Tx-j-a=Nr'P~P'rx-~Nr'P~P(xi-y). [*2.55-564])D. (2[y). xP~y. a =Nr'lP (x~- y). [(1))a.e Nr"lD'(P ~ P*'6r)s (4) F-. (12). (4).) DI- Hp. D. Nr"D'(P P*'x)s ~, [*260535-351])P ~P*'xE o,: D F-.Prop *265-441. I::PecSer. Q, R eC n R1'P.R C Q.)D. PW'OR = P"G'Q.' Q"G'CR =CGQ Dem. F. *263-27. Transp.)-DF: Hp. D.'., E! maxQ'G'R. [*205-123] D.CC'RC Q"CC'R. (1) [*37-2] D. P"IC'IR C P"IQ""C'R [*37-15'2] C P"G'CQ (2) F..*263-47. Transp. D F: Hp.) D. p'Q"c'c= A. [(1).*202501] D. C'Q = Q"'ICR. (3) [*2OlS5.Hp] D. PC"U'Q C P"f'C'R. [('2)] D. P"U'CR = P""C'Q (4) V..(3). (4).)IF. Prop *265-45. F:.Pccol.QC-P:xcC'Q.)Z.-2!Qcx - Pf#I: Qe). S=z tx E 'Q. y = tninQ'(Q 'x - Pf.'x)}. R =8 rS*'B'Q: Rpe co. RPO CQ. P"G'cRPO =P' 'CQ Dern. F. *32 181 FI: Hp)S CQ. (1) [*91-59.*201-18] D. RP.Q(2) F. *263-11. D I:. lip.): xe CQ.)x. E! S'x: [*71-571])Se Cis-I.C'Q CD'S: [(1))Se Cis-*1. PS CDS: [*1 22-51.*962 1] D Re Prog: [*263-1]:) RP E c (3) F-. (2). (3). *265-441. D F - Hp.). P"IC'B = PW'cQ (4) F. (2). (3). (4).) F-. Prop *265-451. H:.Hp*265'-45.):xeC'R.:).P~xI-R1'fx),eK Dem. F. *265o'45. *263-14.) I:. Hp.): xeCGR.). R'x= S'w. [Hp] D E.,xcPl - Pf.'x. [*260-131]:).P(xI-Bi'x),cClsinduct (1) F.*265-4l.)F:.Hp.:):xeC'R.:).P(XI —Rl'x)cE0v Cls induct (2) F-. (1). (2). D F. Prop

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SECTION E] SECTION E] THE SERIES OF ALEPHS,17 177 *265-452. F: Hp*265>45.a!P(xI-R1'x)AiP(yI —RI'y).:).x=y Dem. F. *201-18.) F:. Hp.:xP(R, iy). yP (R,'x): [*14-21])x, y E O'R. xP' (R1'Cy).yP (Ri 'x): [*204-41.*265A45]) xR50 (Ri'y). yRpO (I'x): [*204-71]) x= y. v. xRP y~y= x. V ayRp x: [*4-41]):x = y. v.x.RP yyRp x: [*204-13.*265-45]): x = y:.) F. Prop *265-453. F: Hp *265-45. K = ""{(HX).-X ECR. a=-P (xI~ —RI'x)}. Ke E K0 Cl excl'K0. s'Ic = P"GC'I A P*,"C'I? E*265-451'452] *265-454. F:. Hp*265-453: KEK0 ACl excl'K0.,. S"K ENK0:). P"CI'I % P*"G'CR E No [*265-453] *265-46. F:.I'eehl.QEOARI'I':xecfQ.),.a! Q'rx-Pffl'x: K E NK n Cl excl'K0.). 8"K E6 No: P "'C'Q6 E K [*265-41-454. *123-421] *265-461. F:Hp *265-46.).! p'I"'GQ [*265,46-401] *265-47. F:.I'ecol. QewA~RI'I':KceN~eClexcl'No.:)K s'/e o:). a! p'I"W'rCrQ [*265-461-431] *265-48. F:.KEK0A,Clexcl'N,.)K.s'KteK,:):I'Ewl.QEWARlI'.:).E!ltp'Q [*265-47. *250-123] *265-481. F:Mult ax.D). Hp *265-48 [*113-32.*123'52] *265-49. F:.Multax.:): PE6w,.QEWoARlI'.:).E!ltp'Q [*265-48-481] This proposition shows that, assuming the multiplicative axiom, any progression of ordinals of the second class (i.e. consisting of series having K0 terms) has a limit in the second class, because Ne E,. *265-5. F:I'e oi. QE w.G'Q C C". rE! maxp'C'Q. R = 5~p {xe E" G. y = minQ'(I"x A Q'x)1. S = R r R*'B'Q.) Dem. F. *205-11. F: Hp.).R CP.R CQ.(1 [*201-18] P CI' P. SP, C Q (2) F.*205-197. DF: Hp. x eC'Q Q*'Cx CI*'x.D. x =maxp'Q*'(x (3) F. *263-412. *261-26. DF: Hp. xc4JQ)E! maxp'Q'X (4) R &W III. 12

1'78 SERIES [PART V 4- - F.(3).(4).*205-193.)I:Hp.xeC'Q.Q*'xCP*'lx.).E! maxp'C'Q (5) 4- - F.(5). Transp.)-:. Hp.):xeC'Q.D.a!Q*'x-P*'x. 4- 4 -[*91-542. *202-103] D. 21! Qx n P'x. [*250-121] D.E! R' (6) K.(1).(6).*122-51. DI:Hp.). SeProg. [*263-1]). S to (7) F.(2).(7). *265A441. D F:Hp.. PI'P'Sp = PC'tQ (8) *265-51.:Hp*265-48.Pe olacre K, I C"P. E! maxplla.:).E!Itpla Dem. lF*265-5.)-: FHp.).(S).Se w Rl,'P.PW'S=a (1) F.(1). *265-48.)IF. Prop The following propositions follow easily. *265-52. -:. Hp*265'48.PeW1j.): an G'PeN,v Cls induct. = '!CPnp'Plan G'P) [*265-51A41] *265-53. F::Hp*265A48.D:.Pecw. j=: 4 - P e 11: a n CIP E N, v Cis induct. =.! C'P n p'P"(a n ('P) *265-54. I: P e w,.D V'P C ltp""C"'(c Rl'P) [*265-5] I.e. every limit-point in an wo is the limit of a progression, which is what (following Hausdorff) may be conveniently called an co-limit. *265-55. F:Pecoi..(I pV'P =tp'P"(c Rl,'P) [*26554. *216-602] This proposition does not, like *265A48, assert that every progression in P has a limit, and therefore it does not require the hypothesis of *265-48.

SECTION F. COMPACT SERIES, RATIONAL SERIES, AND CONTINUOUS SERIES. Summary of Section F. A compact series is one in which there is a term between any two, i.e. in which P C P2, where P is the generating relation. We may call any relation P compact when P C P2; then a transitive compact relation will be one for which P = 2. Hence a serial relation P is compact whenever P = P2. Compact series in general have certain properties, some of which have been already proved; but the majority of the interesting propositions in this subject come from adding some other condition besides compactness. Thus series having Dedekindian continuity, which have many important properties, are such as are compact and Dedekindian. Rational series (i.e. such as are ordinally similar to the series of all rational numbers, positive and negative, or, what is equivalent, to the series of rational proper fractions) are defined as such as are compact, without beginning or end, and consisting of oK terms. Such series, also, have many important properties. A continuous series (in Cantor's sense) is a Dedekindian series containing a rational series in such a way that there are terms of the rational series between any two terms of the given series. This species of compact series also has many important properties. It consists of all series ordinally similar to the series of real numbers including 0 and oo. 12-2

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*270. COMPACT SERIES. Summary of *270. The propositions of the present number are mostly either obvious or repetitions of previously proved propositions. The latter are repeated here for convenience of reference. We put comp = P (P G P2) Df, so that the class of compact series is Ser r comp. We have 4- -4 *27011. F:. P ecomp.-: xPy.:),y. a! nP' Py *270'34. F: P e trans n comp.). s'P = sgm'P The proposition s'P* = sgm'P*, which was proved in *212, is a particular case of the above. *270*41. F: P e Ser n comp. 3. Nr'P C Ser n comp I.e. a series which is similar to a compact series is a compact series. *270-56. F:PeSer. Q ef.-E! B'P., E! Q.. P e Ser comp This proposition gives us a means of manufacturing compact series of various types, such as o expr co, co expr (,, etc. *270-01. comp= P (P P2) Df Here "comp" is an abbreviation for "compact." "Compact" series are the same as the series which Cantor calls "tiberall dicht." *270'1. F P comp.. P C P2 [(*270'01)] 4- -4 *270-11. F:. P e comp. -: xPy. D.y.!P'x n P'y [*270-1] *270'12. F: P e comp. -. P e comp [*270'11] *270'13. F: P e trans n comp.. P = P2 [*270-1. *201-1] *270-14. F: Pe Ser n comp.. Pe Rl'J connex.P=P2.. Pe Ser. P= P2 [*270-13] *270-15.: P e Ser n comp.. P e Ser. P1 = A [*201-65. *27014]

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SECTION F] COMPACT SERIES 181 *270-2. F P6Ecomp.3). -'H'! maxp'1P'x [*205-25.*270,1] *270-201. F:P ecomp. ')' 2[! minp'G'"P. -.! maxp'D'P Dem. F. *37'25.) F. MinP'C'P =P"'D'P- (P2) 6D6P () F(1).*270-1.F:Hp.)minp'G'1P =A (2) Similarly F: Hp.:).maxp'ID'P =A (3) F.- (2). (3).:)F. Prop *270-202. F:Pe comp.). a'! minp'P66a.,a! maxp'P 6a [Proof as in *270-201] *270-203. F:Pe~comp.:). a,! seqp't'1x [*206-42. *270-1] *270-204. F:P eSernecomp.E!seqp'a.:). E!maxp'la [*206-451. *270-15] *270-205. F: P e Ser ri cornp. I tp = seqp [*207-1. *270-204] *270-21. F:P eRl',J ncomp. x eC'P.). xltp(P'ax) [*207-31. *270-1] *270-211. F:Pe R'J r, comp.:). D'l1tp = CO'P [*270-21] Thus if a relation is compact and contained in diversity, every member of its field is a limit-point. *270-212. F:Pe connex. D'1tp =CGP.).P ecomp Dem. F.*207-34.:)F:Hp.).CG'P C_-('(Pl. P2). [*33'251]). fl(P _I P2) = A. [*270-1].P ecomp:)D F. Prop *270-22. F:. PcRl'Jn connex.):Pe comp.=. D'ltp =C'P. 2=. PP CD'ltp [*270-211P212. *207-18] *270-23. F:Pecomp- t'A.:).P'-,cBord Dem. F.*270-201.)F: Hp.). (a).a C C'P. a!a.e-a! minp'la. [*250-101]:). P'-,,e6Bord: D F. Prop *270'24. F:Pc Ser ricomp -tiA.). C'P c6Cis induct Demn. F. *270-23. F: Hp.)D. P, 'e fl. [*261'31] D. C'P,e Cls induct::) F. Prop *270-3. F: P c Ser n comp. sect'P - DIP, = P*"C'TP [*211-351. *270-15]

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182 SERIES [PART V *270-31. F: Petransricomp..D,'P,=D'(PPAl) [*211P51.*270-14] -4~ *270-32. F: P e trans r comp. h.Px eDD'(Pe A I) [*211-452. *270-1] *270-321. F: P'P'P C D'(Pe A I).. P comp [*211-451. *2701] *270-322. I:. Pc trans.): P"6C'P C D'(Pe A I).. Pe comp [*270'32-321] *270-33. F:. P eSer.): P e comp..CImaxp d (I'seqp = A [*211P551. *270'14] *270-34. F:Petransncomp.. c'P =Psgm'P [*270-31.(*212-01P02)] *270-35. F-:. Pe trans n connex n comp. P c Ded. PICmaxp = - bseqp [*214'4. *27013] *270'351. F:. P e Ser.: P e comp n Ded.. Pmaxp= -Pseqp [*214'41. *270-14] A series which is compact and Dedekindian is one which has Dedekindian continuity. Thus the above proposition states that a series which has Dedekindian continuity is a series such that every class has either a maximum or a sequent, but not both. *270-352. F: Pc Ser n comp n Ded.a e sect'P.).limaxp'a = liminp'(C7P - a) [*214-42] *270-36. IF: PeRI'Jt comp.:. SpP'P=PP. V'P=P [*2162. *270-211. (*216-05)] *270-4. -: P c comp.. Nr'P C comp Dem. F.K*201-2.)IF: S e P giiQo-.) Q.(S;Q)2 = S;Q2. P =S;Q (1) F.(1). *270-1. D F: P e comp. SeP smor Q.:). S;Q C. S;Q2. [*150-31]) S;;QCS; S;Q2. [*151P252].Q C Q2: D F. Prop *270-401. 1- P c comp. E. Nr'P C comp [*2704. *155-12] *27041. 1- P c Ser comp.). Nr'P C Ser n comp [*270-4.*204-22] *270-411. 1: P e Ser ^ comp. Nor'P C Ser n comp [*27041. *155-12] 4- -*270-42. I:Pecomp. D.P P*'IX, P P*'Xcomp Dem. 4 -F.*270-11.) IFHp.y, z P*'x. yPz..(w). yPw. wPz. 4 -[*90-16] HW) We P).(aw)wcP*'. wPz (1) F-.(1).*270-11.) IF: Hp. D. P C P*'IX comp (2) Similarly 1- Hp. D. P C P*'x c comp (3) F-. (2).(3). D F-. Prop

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SECTION F] COMPAcCT SERIES 183 *270-5. F: P, Q e Ser ^ comp. CGP n CGQ = A e(E! B'P. E i B'Q).). De.Pt e Ser n comp Dem. F.-*160-51.)-: Hp.:). (Pt Q)2 = P2 V Q2 V D'P T C'&Q v C'P TQ'Q [*93-103.Hp] = P2 V w2 w C'P T ClQ (1) F. (1). *270-1. 3 F: Hp.:). P4AQ C (Pt Q)2 (2) I-.(2). *204k5.: F. Prop *270-51. F:PeSer. comp.PC'(P Ser ncomp.PeRe12 excl.). Der. Z'P eSer n comp. *20452.F): HpY.. Z'P e Ser (1) F. *162 1.) JF. (211P) = (h~C6P)2 wj (F;P)2 Vj (h6'CP) I (F;P) w (F; P) I (91'UP) (2) F. *270-1.) F Hp. x (h'G'P) y.. (HQ). Q E C(P. XQ2y. [*41-13]). X (B6CP)2 y (3) F.t*270-1 2)DF HpF. x (F;P)y. x (F;P2) y. [*16312.*20P2]. x (FP) 2 y (4) F. (2). (3). (4~) - *162~11.: F: Hp.:. VP C. (ICP)2 (5) F.(1).(5).) F.Prop The hypothesis of *270-51 is in excess of what is required for the conclusion, which only requires, in place of Peconp, that there should be no two consecutive relations in CGP of which the first has a last term while the second has a first term. This is proved in the following proposition. *270-52. F: P E Ser n Rel2 excl. G'P C Ser n comp. B"~P,1110P n Cnv"(IB)= A. =).:'P E Ser n comp Derm. F. *2701. *163-12.) Hp. 'eCP C (hICCP)2 (1) F.*201-63.) F: Hp. F.F;P = F;P, V FP2 (2) F. *93-103.) F:. Hp. QP,R.:): D'Q = C'Q. v.PGIR = CUR (3) F. (3).) F:. Hp. x (F;P,)y.): (SQ, R): x e D'Q. ye C'R. v. e C'&Q. ye WR: QP1R: [*33-13-131-17]): (HQ, R, z): xQz. z e C'Q. y e C'R. v. x e C'Q. z e C'R. zRy: QP,R: [*150-52. *201i63]): x t"C'P) I (FP)J y. v. *x t(F;P) I (9'C'P)} y: [s162~I] ~~~ X (Y 1P)2 y (~ F.*163-12.*201-2.)F: Hp..F;p2 = (F;P)2 (5) F.(2).(5).*162-1.F:Hp.. F;P C (Z'P) 2 (6) F.(4). (7).*16201.2 Hp. P. ro CP C (SCp)2 (7) F -(4). (7). *204-52.:) F~. Prop

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184 SERIES [PART V *270-521. I:.P E Ser A IRel2 excli. G'P C Ser n comp: C'PACnv"UI'B-A.v. C'PriPB=A:).Y.,PeSerncomp [*270-52] *270-53. I-:PeSer. QeSerncomp.,-..(E!B'Q.E!B'Q).:).PxQeSer n comp Dem. F.*166-1.:)F.PxQ=j'Q~jP (1) F. *165-21.:)F. Q.;PcERel12 excl (2) F.*165-25.*204k21. D -: Hp.f! P. D. Q J; PecSer (3) F. *165-26. *270-4. D-: Hp.:). C'QI;PCSer ncomp (4) F.*151-5.*165-26.:)F:Hp.rE!B'Q CQ~~GBA (5) F. *15P5. *165-26.) I-Hp.e-,E!B'Q.).C'Q 4; PnCnv"G'(fB=A (6) F. (1). (2). (3). (4). (5). (6). *270-521. D F:Hp.ft!P.D. PxQcSerncomp (7) F. *166-13.3 F: P=A.D. P xQE Ser ncomp (8) F. (7). (8).:) F. Prop *270-54. F:PeSerrncomp.r-,E!B'P.xr-..-,eC'P.:).P4*>xeSerncomnp Dem.. F. *204-51)F:Hp.)D. P -I+xe Ser (1) F. *161-1. D F: Hp.D (P+) ~ I ~ [*93-103] =P2 jC6P tIlX (2) F. (2). *270-1. DF:Hp.D. P +>X C.(P _I*X)2 (3) F. (1). (3). DF. Prop *270-541. F:PeSerncornp.rE!B,'P.x,,EC'P.:).x+FPeSerneomp [Proof as in *270-54] *270-55. F:Pef~.CG'PCSer.,-,E!B'P.C'PACnv"U'C[B=A.:). fl1'P e Ser comp Dem. F. *251-3.) F: Hp.:). 17J'P e Ser (1) F. *250-21. *93-103. F: Hp. Qe6C'P. M eF4"C,'P.:). (ajx). (M'lP1'Q) (P'Q) X (2) F. *200-43.)D F: Hp(2).(M'Pi'Q) (P1'Q)x. L=Mr(-t'Pi'Q)ewx 4,(P1'Q).:).M(lII'P)L (3) F. *200-43.)D F: Hp(3). N~e F'C,'P. (M'Q) Q(N,'Q). MrP'Q= NrP'(Q.).L(lI'cP)N (4) F. (2). (3). (4).)D — F:Hp.M, NEF4'C'P.QeC'P.(M'Q)Q(N'Q).MrPi(Q=NrP(~Q.:). (HL)..M(H,'P)L..L (1'"P) N (5) F. (5).*200-43.:)F:Hp.:).IJ'PC.(H-lP)2 (6) F.(1).(6).) F. Prop

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SECTION F] COMPACT SERIES 185 *270-56. F: PSer.Qen. - E B'P.r E!B'Q.). PQeSern comp Dem. I. *176'151. D I: P = A.).P Ser n comp (1) h. 176-181-182.) I. PQ smor IH'P;Q (2) *. 165-25. 51'121.) H: Hp.! P.). P;Q e (3) F. 165-26. 204-21.:: Hp.. C'P;Q CSer (4) H.*165-25.*151'5.:)F:Hp.!P.). E!B'Cnv'P,;Q (5). *165'26. *1515.) F-: Hp. 2. C'P J;Q n Cnv"'B = A (6) -. (3). (4). (5). (6). *270.55.) I: Hp.! P.). nP J,;Q e Ser n comp. [(2).*270-41]). PQ e Ser n comp (7) F. (1). (7).) F. Prop By means of the above proposition, compact series can be manufactured by taking series of such types as c expr w, o expr oi, cw expr w, etc. Any power aexpr / consists of compact series, if 8/ is an ordinal having no immediate predecessor, and a is any serial number having no immediate predecessor (i.e. not formed by adding i to a serial number).

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*271. MEDIAN CLASSES IN SERIES. Summary of *271. We shall call a class a a " median" class in P if a C C'P and there is a member of a between any two terms of which one has the relation P to the other. When this is the case, we have xPy. X,y ((z). z e a. xP. zPy, i.e. PCPralP. Thus P cannot contain any median class unless P is compact. Conversely, if P is compact, C'P is a median class. Hence relations containing median classes are the same as compact relations. Median classes are important in dealing with rational and continuous series: the rationals are a median class in the series of real numbers, and the series which Cantor calls continuous are characterized by the fact that, in addition to being Dedekindian, they contain a median class which forms a series of the same type as the rationals. If P is a compact series, the class P"(i'P is a median class in the series s'P (*27131). This fact is used in proving that the series of segments of a rational series is a continuous series. Our definition is med=&P(aCC'P.PCPraIP) Df. Thus med'P will be the median classes of P, and "P e a'med" means that there are median classes of P. We have (P'med = comp (*271*18); also *271-15. F: amedP.. P, P t a e comp *271'16. F: (a n C'P) med P. =. (a n D'P) med P. =-. (a (I'P) med P.. (an D'P n('P)medP If P is a series, and a C C'P, a is a median class when, and only when, its derivative is (IP, i.e. *271'2. H:. P e Ser. a C C'P. D): a med P. -. (PP = 8p'a An important proposition is *271'39.: P. QeSern Ded.a med P., med Q. (P a) smor (Q, /3).. Psmor Q I.e. if P and Q are Dedekindian series, and a, 3 are median classes of P and Q respectively, then if P a and Q,8 are similar, so are P and Q. This

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 proposition is proved by showing that P is similar to the series of segments of P ~ a, the correlator being Itp with its converse domain limited (*271-37). Another important proposition is *271-4.: Se P smo- Q. 3 medQ.. (S"/) medP I.e. a correlator of P with Q correlates median classes with median classes. The above two propositions are used in *275'3'31, which prove that two series which are continuous (in Cantor's sense) are similar, and that a series similar to a continuous series is continuous. *271-01. med =P (caC'P.P PralIP) Df *2711.:. amed P. a C C'P. P Pra P:: a C (CP:xPy. 3)y. g!a P'x ^ P'y [(*271-01)] *271-11. F: a med P. =. a med P [*2711] *271-13.: a medP. /C C'P.). (a u,) med P [*271'1] *271 14.: a med P. D. C'P m a med (P C a) Dem. F. *2711.) F:. a med P.): x, y e a. xPy.),.y. (az). z e a. xPz. zPy. [*35-102],yY (3z). za.x(P a)z. z(Pa)y: [*35'102.*271-1]): C'P a med (P c a):. D F. Prop *271-15. F: a med P..P, P a e comp Dem. F. *271-1.: Hp.. P C P2. [*270-1]). P e comp (1) -. (1). *271-14. D F: Hp.. P a e comp (2) I-.(1).(2).).Prop *271-16. F:(a n CP) med P. -.(a n D'P) med P. -. (a n I'P) med P. (an D'Pn Ci'P)medP Dem. F. *271-1. *33-15. 4- -- F:. (a n CP) med P.: Py.),,,y.! a n D'P ^ P'xn n P'y: [*271-1] -: (a n D'P) med P (1) F. *2711. *33-151. D F: (a n C'P) med P. =(a n (IP) med P (2) F. *2711. *33'15151. 3 4- -+ F:. (a n C'P) med P. -: Py.),y. a I a n D'P n (I'P ^ P'x n P'y: [*271-1] -: (a ^ D'P n I'P) med P (3) F. (1). (2). (3) F. Prop

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188 SERIES [PART V *271-17. 1- P ecomp.:. C'P, D'P,GQ'P em~ed'P Dem. F.*35-452.*2701I.: I-Pe comp. 2. PC- P (G'P P. [*2711]).U'~~~~~GIP emed'P. (1) [*271-13].C'IP emed'P. (2) [*271-16] 2.D'Pe med'P (3) F-.(1). (2). (3). 21-. Prop *271-18. F. Pin1ed = comp [*271-15-17] *271-2. F:.PeSer.aCC,'P.2):amedP.=-. PIP=Sp'la [*216-13.*271,1] *271'3. F-: P e RI' n trans. a med P. 2. P"a med (s'P) Dem. F. *271-15. *270-34. DF: Hp. D. s'P =sgm'"P. [*212-11]2.'P,?/,yD(eI<iy-3 (1 [*37-1] D.(ax, y).-x ey -/3.xPy -y y. [*271-1] 2. (ax, y, z). a c ry -/3. xPz. zPy. z c a. ye Lry. [*201-12] 2(ax,Y, z). xcr y -3.wXPz. zPy. zca. Ycery - -.(yPz). [*32-18] D. (az). zcea. a! P~z -/3.aI y — P'z. [(1).*270'322] 2.(az). z c a./3 (lg'P) (P'Z). (P'z) (S'P) Cy (2) F. (2). *27 -1.2 F. Prop *271-31. F:P cRl'J ntrans n omp. 2. P"G('P med (cP) [*271P317] The following propositions lead up to the proposition *271'37. F:PcSerADed.amedP.2).ltprC's'(P~a)cEPsmorts'(P~a)} whence, if a is a median class of P, P is similar to the series of segments of P ~ a. This proposition is used in proving that every continuous series is similar to the series of segments of a rational series. *271-32. F: PeSer.R=P~ a. /eD'Re.El ltp'/3. 2. /3=R"/38=a ~P'ltp'/3 Dem. F. *205-9.2D F: Hp. a nC,'P r,, cI. 2. maxj/31, = maxp (a n /3) [*37A413.*211-11] = maxp'/3 [*207-13] = A (1) F. (1). *200'35. 2 F: Hp. 2. mnaxR'/ = A. [*211-42-12] D2./3= R"/3 (2) F. *207 -231. DF: Hp.D. P"/38= P'ltp'/3. [*37-413] 2.R"/3 = a vn P'ltp'/38 (3) F -. (2). (3). 2F. Prop

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SECTION F] MEDIAN CLASSES IN SERIES 189 *2071-321. F:P cSer.R =P ta.2. tp r D'IRee I-+,1 Dem. I -. *271[32. 2) I Hp. /3, 7ye D'Rc. Itp'/3 = Itp"7. 2. / = fy 2 1- Prop *271-322. F:PE Ser.R P.2.altp; 'RT CP Dem. F -. *212-23.2F:. Hp. 2 x (ltp;'91R)y. (H/,8,ey). /3,y E D'Re../3 C~. /3 + y. x= Itp'/3. y =ltp~f7. [*207'2:31] 2(a/3,,y)./3, eyD'Re./3 Cy./ + y. P'x= P"/8. P'y=P"7f. [*37-2. *271i321] 2.P'x C P'y. y. [*2O4-33] 2.xPy: 2F. Prop *271-33. F:P etrans.a med P2P'x = P"(anAP'x) Dem. F. *201-501.2F:Hp. P"CP'x C P'x [*37-2] 2P"1(a P'Px) C P'X (1) F. *271 '1. 2F:.Hp.2:yPx.2.(az).yPz.zea.zPx. [*37-1] 2.y e n11(P'x) (2) F. (1). (2). 2 F. Prop *271-331. F: Hp *271.33. R= P ~a..a AP'x-=R"I(a ri P'lx) Dem. F. *271P33. 2 F Hp. 2. a n P'x = a A P"C(a n P'x) [*37-413] = R"I(a ^ P'x): 2 F. Prop *271-332. F:P cSer a med P. xEG'P 2x-=Itp'r(anPCX) Dem. F. *271'331.2 F: Hp. 2.ar i PX C P"C(a nP'x). [*205-123] 2.maxp'(a n% P'x) =A(1 F: Hp. 2. x e C7'P. P'x= P"11(a r^ P'lx).. E! maxp'(a n P'x). [*207 521] 2. x=Itp'(a v^ P'x): 2 F. Prop *271-34. F:PeSer.amedP.2).P=1tP;S'(P~a) Dem. F.*271-331.*211-11.2 F:

Hp.R=Pta. 2. anP'xeD'Re (1) F.*204-33. 2F:Hp.xPy.2).anP'xCaznP'y (2) F. *271P332.2F:Hp.xPy.2).x=ltp'(anP'Ix).y=ltp'(anP'ly). (3) [*204-1] 2a ^PCX +a eP'ly (4)

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190 SERIES [PART V F-. (1). (2). (4). *212-23. 2 F. (3). (5).2DF:. Hp.D: xPy)x [ltp~s'(P at)}y (6) F. (6). *271P322. DF. Prop *271-35. F:amedP. 2.D'(P~a)eC-Q'~maxp Dem. F.*37 413.*21111.2I [*37-1]:): (Hp): xe/3 2:z-(HY).Ye pnA e xPy (2) F. (2). *271,1.2 F:.Hp. /e ED'(P a)c. 2): (as): x e/3 2,,. (ajy,z). xPz. Za.- zPY. Ye pn F'. [(1) 2)~'. (aZ). xPz. z E /3 [*371] 2,,~~~~DX. x CP"/38 (3) F. (3). *205 123. 2 F:Hp.8 /3 D(P a)e. 2. maxp'/3 = A: 2 F. Prop *271-36. F:P eDed.a med P. 2. D'(P ~a), C Pltp [*271P35.*214'101] *271-37. F:PESernDed.amedP.2). ltprC's'(P~a)EPsmor{s'(P~a)} [*271V32P34-36. *151P22] *271'38. F:P eSerr Ded. amed P. 2. P smor {;'(P a)} [*271-37] *271-39. F: P, QESer nDed. amed P./3 med Q. (P ~a)smor (Q ~3). D. P smor Q Dem. F. *212-72. 2) F: Hp. 2D. {;'(P ~ a)} smor {r'"(P t /3)} (1) F.*271P38. DF:Hp.D. P smor{f,;(P ta)}. Qsmor fs'(Q ~/3)J (2) F. (1).(2). D F. Prop This proposition is used in proving that all continuous series are similar, by means of the fact that such series contain rational series as medians, and that all rational series are similar. *271A4. F: SePsmorQ./3medQ.D.(S"/3,)medP Dem. F. *35-354. *7 414.2D F: Hp. 2. Q r,8 /3 = Q I S r S"/3,8. [*150-1] 2; (Q r' /3) = (5;Q) r S"C/3. F. *72-6. 'D F: Hp.D. (Q r/3)I SI'S =Q r3. [*150.1] 2. {S;(Qr/3)} j(S;Q)= SIQr/3IQjS (2) F. (2). *271-1. D F: Hp.D. SI Q I S CtS(Q r/3)}I (SQ). [*151-11.(1)] 2D. P C. (P r S"/31) I P. [*271-1] 2. (S"/38) med P: 2 F. Prop

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*272. SIMILARITY OF POSITION. Summary of *272. If P, Q are two serial relations, and T is a correlator which correlates some terms of C'P with some terms of C'Q, we say that two terms x and y, of which x belongs to C'P and y to C'Q, have similar positions with respect to T if y comes after the correlates of all members of D'T which x comes after, and y comes before the correlates of all members of D'T which x comes before. This notion is useful for inductive definitions of correlations. If we start by correlating any two terms x1, yi, and take another term x, coming (say) after xi, a term Y2 having similarity of position with respect to xi 4, yi must come after yi. Suppose now we take x, between x1 and x2. Then a term y3 having similarity of position with respect to xi, yi V x2 I Y2 must come between y, and y,; and so on. A correlation T constructed in this way will be such that T;Q C P. T;P C Q. If the whole of C'P and G'Q can be obtained by prolonging the construction long enough, T will at last become a correlator of P and Q. This is the principle of Cantor's proof that any two rational series are similar. As a rule, when the notion of similarity of position is useful, the relation T will be one-one, but this is not assumed in the definition. We write " xTpQoy" for "x and y have similar positions in P and Q respectively with respect to T," or, as we may express it more shortly, " the P-position of x is T-similar to the Q-position of y." The definition is TpQ = {x e C'P. y e C'Q. D'T n P' C T"Q'y. D'T n P'x C T"Q'y. D'Tn t'x T'y} Df. This definition states that the predecessors of x which have T-correlates are to be correlated with predecessors of y, the successors of x which have T-correlates are to be correlated with successors of y, and if x itself has a T-correlate, y is to be a T-correlate of x. When T is a many-one relation, the definition becomes somewhat simpler. We then have *272-13. H:: Te Cls - 1.):. xTpQy. -: eC'P.yeC 'Q:zeDD'T P'x. z. T'zQy: zeD'Tn P'x. z. yQT'z: xe D'T..y =T'x

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192 SERIES [PART V We have *27216. F. (D'T)1 TpQ T That is, a term which has a correlate cannot have similarity of position with any term except one with which it is correlated. A member of C'P n D'T will have similarity of position with its correlate (assuming TeCls —l) if P D'T C T;Q. T"C'P C C'Q (*272'18). Under ordinary circumstances, a term which is not a member of D'T cannot have similarity of position with any member of ('T (*272'2). When T is many-one and its domain is contained in C'P, and P and Q are series, and x has no T-correlate, we have (*272'21) TpQy. x: x e C'P. y e C'Q: z e D'T n P'x. -. T'zQy, i.e. in this case, x and y have similar positions if the predecessors of x which have correlates are the terms whose correlates precede y. In this case, if xeC'P, we have (*272'212) 4 — - -- 4- 4-,TPQx = C'Q rN 9 (D'T n P'x = T"Q'y) = C'Q n r (D'T n Px = T"Q'y). We next investigate the condition for C'P=D'TpQ, i.e. the condition required in order that every member of C'P may have similarity of position with some member of C'Q. A sufficient condition is P, Q e Ser. Q e comp. T e Cls —. D'Te Cls induct. P t D'T C T;Q. a! Q. T"C'P C D'Q 'Q as is proved in *272'34. We next consider the reversibility of TPQ, i.e. the condition that the converse of TPQ should be (T)Qp. A sufficient condition is P,QeSer.Tel —1.D'TCC'P. I'TC 'Q (*272-42). Finally, we have two propositions on the addition of another couple x 4 y to T. With the above-mentioned hypothesis of *272'42, if xT/pQy and T;Q C P, putting W= Twx, y, we shall have P D'W = W;Q (*272-51), so that the hypothesis we had for T still holds for W. The propositions of this number are in the nature of lemmas. for Cantor's proof that any two rational series are similar, which is given in *273. 272'01. TPQ = x e CP. y CQ. D'T n xC T"Q'y. D'T n P'x C TQ'y. D'T n 'x C T'y} Df *2721. F:r xTpQy. e CP. y e C'Q. D'T n P' C T"Q'y. D'T ^ P'x C T"Q'y. D'T n 'x CT'y [(*272-01)]

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SECTION F] SIMILARITY OF POSITION 193 *272-11. I-:xEU'P.-). TpQx-CQ r~pQ"'~17"(D'T A P'x) A Afc(T(' P'x) A p'T"(D'TAr t'lx) Dern. F.- *272-1 F: Hp TpQ'x= G'Q n- [z ED'TA P'x).zTj Qy: zeD'TA P'x.),. zTj Qy: ze D'TAn tfx.) D zTy} [*40-51-53] =CG'Q np'Q"'cT"(D'TnAP'x) A p'Q"cT"(D'T AP'x) A p'cT" (D'ITrA t'x)) F. Prop *272-111. F: xEG'P.) TPQ X = C'Q A p'{Q"'cT"(D'TAn P'X) v

University of Michigan Historical Math Collection: Principia mathematica, by Alfred North Whitehead ... and Bertrand Russell.

Q""T"(D'TAn P'x) v T"c(D'TAn t'x)} [*272-11 - *40-18] *272-12. F::xTpQy.E=:. xECG'P.yeC'Q:.zeD'T.)Z:zPx.:).zTlQy: zPx.).zTjQy:z=x.).zTy [*2721-] *272-13. F:: TeCls —+1.):.xTpQy.=-:x6C'P.yEO',Q: zEcD'TAn P'x.)D, T'zQy: cDcT n P'x.)-yQT'z: xe D'T.) y = T'x [*272-1 2 - *71-701] *272-131. F:TcCls —1.xe6C'P.:). [IpQ'X = C'Q A p'tQtT"cP'x u- Q"'T"sPx v T"(D'T A Il~ [*272-111. *71-613] *272-14. F:xeC'P-D'TJ:). JPQ X =C!'Q A% p'Q"~'T"(D'T A P'x) A p'Q"'T""(D'T A P'fx) [*272-111 - *40-18] *272-141. F:xe CYP-D'T). TPQ'x = (C'Q A (D'T A P'x C T"cQ'y. D'T A P'x C T"Q'y)I [*272-1] *272-15. F:T eCis — I. xc G'P -D'T.) TPQ X = GQ Ai p'Q"cT"P'x A p'Q"~I7"P'x [*272-131 -*40-18] *272-16. F.- (D'T) 1TpQ C T Dem. F. *272-12. - F: xE D'T xTpQy.)xTy: F.- Prop R.& W. IlL. 13

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194 SERIES [PART V *2724161. F:TEcChs -*I. P ~D'TGCT;Q -D.)(D'Tl)1 TpQ =4IP 1T rC'Q Dem. F. *150-41.)I-: Hp. zeD'T1'zPx. xTy.)D.T'zQy (1) P. *150-41. DIF: z eD'T. xPz.xTy..yQT'z (2) F.(1). (2). *27213 -DF: Hp.xTy. xeC'P ye C'Q D.). xTpQy (3) F.(3). *272-16:)F. Prop *2724-7. F:Te Cls — 1. P D'T C TIIQ. D'TC C'P. VP' C ('Q. T==(D'T)1lTpQ [*272-161] The hypothesis of *272-17 is satisfied in all the important uses of Tp)Q. *272-171. F:Hp *272-17. x ED'T.). TpQ'x = tT'x [*272-17] *272-18. F:TECls —*1.P~D'TC-TQ.T"C'IPCCO'Q.xEC'1-'AD'T.:). Dern. F. *37-61.)F:Hp.).T'xEC'Q (3) F - (1).(2). (3).-*27 213. D F: Hp.) D.x'pQ(T'x) (4) F. *272-13.:)F:Hp-xTpQy.).y=I 'x (5) F. (4). (5).) F. Prop *272-2. F: TE Cls-*1.,. D'TCGIP Pe connex. Q CJ. x,c D'T.) TpQ'x n WI'= A Dem. F.*272-13. DF:Hp. xTpQy. ze D'T nPcx. D. I'z y (1) F.*272-13 D)F:Hp.'xTpQy. z eD'TAriP'x.).T'zz (2) F. (1). (2) D) F:Hp. xTpQy - z E D'T. D. -7, T'zy: D F. Pi-op *272-201. F: TeCls — 1. D'TCCG'P.PEcon~nex.-! D'TpQ- DT.:). PrT C G' Q Dem. F.*202-104.)F:.Hp. zED'TaxTpQy.xcsED'T.):zPx.v.xPz: [*272-13]): T'zQy. v.yQ (T'z) [*33-132] D: T'z cG'Q:. DF. Prop *272-21. F:: T cCls-*1I. D'TCCY'. P, QeSer. x- seD'T.):D xTpQ~ y. x ie O'P y E G'Q: z e D'T ri P'x. —, T'zQy Dem. F. *272-2. D)F:. Hp. zEcD'T. xTpQy. D: xjzz. y +T'Iz [*204-3.*272-201]): xPZ r.. (zPx): yQ (T'z) - r. (T'z) Qy} (1)

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SECTION F] SECTION F] SIMILARITY OF POSITION15 19.5 x~EC'P.y~EC'Q:. zED'T.),:zPx.:).T'zQy:c-(zPx).).,,(T'lz)Qy (2) F-. (2).)D F:: Hp.):. xTpQy. =xe G'P. yE C'Q: z ED'T. zPx. -z. T'CzQy::) F. Prop *272-211. F::Hp*272-2l.D):. xTpQy. xeC'IP.yEC'Q:zeD'TAP'x.=E-z.yQ(T'z) [Proof asin *272-21] *272-212. F:Hp *272-21.XE G'P.). TPQ'x = G'Q rA ^ (D'T A P'x = T"cQ'y) = (J'Q A 9 (D'T A P'x - TP'Q'y [*272-21-211] *272-22. F:TECls —+I.P,QEtrans.xTpQy.z,wED'T.xEP(z-w).:). Dem. ~~~~~~~~~~y EQ (T'z - T'w) F.*27 2 13.) F: Hp.)T'zQy.yQT'w:)F.Prop *272-221. F: TCls -+ 1.P, Q ctrans. D'TpQAr P (z -w). (T'z) Q (T'w) [*272-22] *272-23. F:. TE Cls -+ I P, Q c trans: z (P ~ D'T) w.z)w D!DTpQ A P (z -w):) P D'T C T; Q Dem. [*150-41] D. z(T;Q)w:.)DF. Prop *272-24. F: D'TA G'1P=A.). TpQ= C'P TC'Q [*272-1] *272-3. F:TEC1s —+1.SC-T.).TpQC-SpQ Dem. F. *272-13. DF:.

Hp. xTpQy.D: zEcD'T. zPx.) T'zQy: [*72-9])z ED'S. zPx.).S'zQy (1) Similarly F:.Hp. xTpQ y.)z ED'S. xPz.)yQS'z (2) F. *272-13.)DF:. Hp. xTpQy.): zED'T. z= x.)D T'z- =y: [*72-9]): zeD'S. z= x.D.S'z = y (3) F. (1). (2). (3). *272-13.)DF: Hp. xTpQy.) xSpQy:) F. Prop The following propositions lead up to *272-34. *272-31. F:P,QESer.TECls-*1I.x-,ED'T.z=maxp'(D'TAP'x). 4- 4- %. W = Minp'(D'T A P'x). P ~ D'T C T;Q.).TpQ'x = Q (T'1z - T'w) Dem. F.*205'-)21.:)F: Hp.uED'TnP'x-ffz.).uPz. [*150O4l.Hp] D. T'uQT'1z(1 13-2

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196 SERIES [PART V -().I: Hp. yEQ (T~z - T'w). uED'T P'x.)D. T'uQy (2) Similarly F: Hp. ye Q (T'z- T'w). uED'T ' P'Ix.). yQ Tu (:3) F-.(2). (3). *272-13.)D-: Hp. yEQ (T~z - Tw).). xTpQy (4) F.*272-22. DF: Hp. D.TpQ'x CQ (T'z - T'w) (5) 1-. (4). (5). D F-. Prop *272-32. -: P, QeSer. TcCls->. D'TCP'x. P ~ D'T CE T;Q. = mnaxP'D'T.). TPQ'x =Q'T'z Dem. F. *272i13. DF::lHp.D:. xTpQy.~:ucD'T.:)U, IT'uQy (1) F-. *205-21. D I: ip.? c D'T - tcz.). uPz. [*150-4l.Hp] D. T'uQT'z (2) [(1)]: xTpQ y (3) F. (1). D)F:llp. xTpQy.D. T'zQy (4) F-. (3). (4). D F-. Prop *272-321. F:PQESer.TECls-*1,.D'ITCP'x. P ~D'IT C]IQ. w= minp'D'T.:). TPQx = Q'T'z [Proof as in *272-32] *272-33. F:P, Q eSer. Q Ecomp. Te Cls-*. D'TE CIsinduct. Dem. ~ P ~ D'TEC T;Q.). (P"cD'TAr P"'D'T) - D'T C D'TpQ F. *261-26.)DIF:Hp. a! D'T nP'x.D. E! mirip'(D'T nP'x) (2) F.*205-1 1-1 11.)D F: Hp - x - EDT. z= maxp'(D'Tr-i P"'x). wv= w inp'(D'T n P'x).). zPw. [*150-41]). T'CzQT'W. [*270-11]). a! Q (T'z - T'1w). [*272-31] D. g!TpQ'x (3) F: Hp. x rE D,'T. a I DT nP'x. a "! DT nP'x. D. xE D' TPQ: D F. Prop

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SECTION F] SIMILARITY OF POSITION 197 *272-331. F: Hp *272-33. 4! Q. T", C'P C D'Q.:).CG'PAp'P"cDTC D'TpQ Dem. F.*261-26.:)F:Hp.2!D'TriC,'P.:).E! rnaxp'D'IT (1) +- +- 4-~~~~~~~ F.- *272-32.) F: Hp. x ep'P"'D'T. z = maxp'D'T.:). TpQ'x = Q'T'z. [*33-4] D.a! TPQ'x (2) F.*35-85.*272-24.) F: Hp. D,'TniC'P=A.).C'PCD'TPQ (4) F. (3). (4).) F. Prop *272-332. F:H1p*272-33. j! Q. T"G'lP CU'Q.:).C'Penp'P"(D'TCD'(TpQ [Proof as in *272-331] *272-34. F: Hp*2721.'3.ftQ.TI"C'PCD'QnPlQ.D.C,'P=D'TpQ [*272-33-331-332-18. *202-505] The following propositions are lemmas for *272-42. *272-4. F: P, Q eSer. TE 1 — +1.D'T CCG'P.PT CC'Q. x Je D'T. xTpQy. y(T)Qpx Dem. F. *272-21.) F:. lip. x E C'P - y E C'Q: zE D'Tn P'x. ~-T'zQy: [*72-243]: xEGCY. yE6C'Q: (T'w) Px. ~w e 'T -wQy: [*272-21] D: ly(T)Qpx:.) F. Prop *272-41. F:P, Q ESer. TE I — 1.D'TC CY'P. TCGcQ. xEcD'T.wxTpQy.).y(T)Qpx Dern. F. *272-13. F:-: Hp.:):. XE COP. y =T'x: zED'TAnP'x.)Z.T'zQy: z eD'TAnP'x.)Z,.yQ(T'z):. [*204-3]):. x ECP. y =T'x: z eDc'TAP'x.)Z T'zQy: z~ED'T - t"x -P'x).D T'z j y c.- { (T'z) Qy} [Transp]): x XE G'P, y= T'x:. z E D'T- t'x. D: zPx. =.(T'z)Qy:. [*720-24]):ECYP.y-=T'x:. (T'w)Px zx. EZ.WEUT*W)Qy:. [*71-2362) x yEG'Qx- T'y-: (T'w) Px. =Ez. w EP(T. wQy: [*14-21.*33-43])D:. yE C'Q. x = T'y:. we U'11T.),,,: (T'w)Px..wQy:. [*204-3]):.y EC'IQ.x= T'y: -wEE'IITA Q1y. D,. T'wPx: wEUIT A Q1y. %. xP(T'~w):. [*272-13]

[*-27213]:.y(T)Qpx:: DF. Prop

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198 SERIES [PART V *272-42. F:P,QESer. TE 1-*1I.D'TCC'P. APTCC'Q.).(T) Qp=TPQ Dern. F.*272-44I.:)F:Hp.:).TPQC(T)QP (1) TTp F-. (1).(2).) F.Prop *272-43. F:P,QeSermcomp-t'Ak.Te1I —+1.D'ITCD'PrdI'P. C'T C D'Q A[PQ.P ~D'T= T;Q. D'TeCls induct.:). D'TpQ =OP. EJI'TpQ = C'Q Dem. F*272-34.)D:Hp.:). D'TpQ =O'P (1) F. *150-36.)FD T;Q = T;Q c1'T. T;P=T;P ~D'T (2) [(2)] =T;P (3) F. *1 20-214. D F: Hp.:). ELT eCls induct (4) F. (:3). (4). *27234. D F: Hp.)D. C'4Q = D'(T)Qp [*272-42] =J'PQ (5) F. (1). (5).)DF. Prop *272-5. F:P, Q cSer. TE Cis -+1. D'T CG'P.xTpQy. TQ CP.) (T~ 4 y);Q C P Dem. F*150-75. D F. *272-212.:)F: Hp.x,,eD'T.:). T"Q'yCP'x. TI"QyCP'x (2) F.*272-16.)F:xe6D'T.D.Tix4jy=T (4) F. (3).(4). D F. Prop *272-51. F:P,QeSer. Tel-+1.D'TC CP.G'ITCO'Q. xTpQ~y. P~D'IT=TIQ. W=Tvxjy.).P~D'IW= W;Q Dem. F. *272-5. DF: Hp.D. W;Q CP (1) F. *272-42. D)F: Hp.:).y(T)Qpx (2) F. *150:3'6. *151P26.) F ~:Hp. D. TP =Q ~ 1'T (3) F.(2).(3).*272-5.)F:Hp.).W;PCQ (4) [*I 51-26] D. P ~D' W W; Q: DF.Prop

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*273. RATIONAL SERIES. Summary of *273. A "rational series " is a series ordinally similar to the series of all positive and negative rational numbers in order of magnitude, or, what is equivalent, a series ordinally similar to the series of all rational proper fractions (0 excluded). This characteristic of rational series is not, however, the most convenient for purposes of definition. Following Cantor, we define a rational series as one which is compact, has no beginning or end, and has R0 terms in its field. Thus the field of a rational series can be arranged in a progression, and this is the source of the special properties which distinguish rational series from other compact series. Rational proper fractions can be arranged in a progression in many ways, for example the following: If two fractions (in their lowest terms) have the same denominator, put the one with the smaller numerator first; if they have different denominators, put the one with the smaller denominator first. We thus obtain the series 1 1 2 I 3 1 2 3 4 1 2, 3, 3, 4, 4X, 5' 5, 5, 5',.. This series is a progression, and contains all rational proper fractions. Conversely, the natural numbers can be arranged in a rational series. Take, e.g., the following arrangement: Express the numbers in the dyadic scale, so that every number is of the form E 2" (e K), where K is a finite class of integers. The relation of the number to K is one-one. Arrange the various K's by the principle of first differences, i.e. form the series M1 C (CIs induct - t'A), where M is the relation "less than" among finite integers. The resulting series is a rational series; thus the integers are arranged in a rational series by virtue of their correlation with the classes K. This arrangement places all the odd numbers before all the even numbers, all numbers of the form 4v+2 before all numbers of the form 4v, and so on. If two numbers are expressed in the dyadic scale, their relative position in the series is determined by the first digit (starting from the right) which is not the same in the two numbers: the one in which this digit is 1 precedes the one in which it is 0.

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200 SERIES [PART V The two chief propositions in regard to rational series are (1) that any two rational series are ordinally similar, (2) that if R is a progression, its finite existent sub-classes arranged by the principle of first differences form a rational series. The second of these propositions is proved by showing (a) that the finite existent sub-classes arranged by first differences form a compact series, (b) that the finite existent sub-classes arranged by last differences form a progression. By this means, given any progression, we can specify a relation which arranges its terms in a rational series. For if T is a correlator of our progression R with the progression Rie: (Cls induct - t'A), then T;Rcl C (Cls induct - tiA) is a rational series whose field is C'R. Hence rational series exist in any type in which progressions exist. The arrangement of the finite sub-classes of a progression, with the resultant existence-theorem for rational series, will be dealt with in the following number. In the present number, we shall be concerned with the proof that any two rational series are ordinally similar. The proof of the similarity of any two rational series is due to Cantor. It is long and rather complicated; in outline, it is as follows. Let P, Q be two rational series, and R, S two progressions whose fields are C'P and C'Q respectively. Construct a series of correlations of parts of P with parts of Q on the following plan: Begin with A, and if T is any correlation, let the next be T seqR'D'T, mins'TpQ'seqR'D'T. Then the sum of all the correlations generated from A by this law of succession will be a correlation of P with Q. It will be seen that, if we put W = XT IX = seqI'D'T, min'TpQ'seqR'DT}, the relation which is to be shown to be a correlator of P and Q is WA, in the sense defined in *259. Thus we have to prove WA I --. I WA = C'Q P= WA;Q. WA e 1 -- 1 results immediately from *259*15. P C D'WA = WA;Q results immediately from *259-16 and *272-51. Thus it remains to prove D' WA = C'P. (I'WA = C'Q. D'WA = C'P is easily proved. By induction, if T is one of the series of partial correlators, D'T e Cls induct, and therefore E! seqR'D'T, by *263'47, 4 -and by *272'34, C'P=D'TPQ; hence [! TPQ'seqR'D'T, and therefore, by 250121 E! mi seq. Hence T has a successor, which correlates *250'121, E! mins'TpQ'seqn'D'T. Hence T has a successor, which correlates

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SECTION F] RATIONAL SERIES 201 seqR'D'T with mins'TpQ'seqq'D'T. Hence the successor, in R, of every member of C'R which has a correlate, has a correlate; hence by induction every member of C'R (i.e. of C'P) has a correlate. Hence D' WA = C'P. The proof of (' WA = C'Q is more difficult. As before, let T be one of the series of partial correlators. We have to prove that there is a correlator which has seqs'(I'T in its converse domain; when this is proved, the result follows by induction. To prove this, put x = minR'TpQ'seqs'GaT. x exists, in virtue of *272-

43. Also since D' WA = C'P, it follows from *259'13 that there is a partial correlator U such that x = seqR'D' U. We then have to prove seqs'('T = mins' UpQ'x. —) —) 4 — Put y= seqs'C'T. Then S'y C C'T. Hence, by *272-2, S'y ro UPQ'x = A. 4 -Thus if xUpQy, it follows that y= min' UpQx. To prove XUpQy, observe that TCGU. UPQ CTpQ. P DU= U;Q. We have u e D' U.. (uTpQy), by *272-2. Hence, by the definition of TPQ, we have, if u e D' U, (I). z e D'T. zPu. (T'zQy). v. (gz). z e D'T. uPz. (yQT'z). In the first case, we have ([z). z eD'T. zPu. (zPx), because xTPQy Hence, since x z because x,, eD'T, (3z).ze D'T.zPu. xPz. Similarly, in the second case, (z). ze D'T. uPz. zPx. The second case is incompatible with xPu, and the first with uPx. Hence xPu.. (). z e D'T. xPz. zPu: uPx.. (z). z e D'T. uPz. zPx. But, since xTpQy, xPz.. yQ (T'z). D. yQ (U'z), because T C U, and since PD'U= U;Q, zPu.. (U'z)Q(U'u). Hence xPu.:).yQ(U'u), and similarly uPx.). (U'u)Qy. Hence xUpQy. Hence y = mins' UpQ'x, and therefore y belongs to the converse domain of the next correlator after U. Hence every term of C'Q belongs to the converse domain of some correlator, and therefore to (1'WA. Hence WA correlates P and Q, and P and Q are ordinally similar.

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 *27301. q = Ser n compn CA n P (D'P = GP) Df Following Cantor, we use qj for the class of rational series. 4 -*273'02. RspQfT = T v seqRDT 4 mins'TpQ'seqf'D'T Dft [*273] *273-03. (RS)pQ = (RSpQ)*'A Dft [*273] *273-04. TRSpQ = 8(RS)pQ Dft [*273] TRSPQ will be shown to be a correlator of P with Q when P and Q are rational series, and 1? and S are progressions whose fields are C'P and C'Q respectively. *2731. F:Pef..=PeSerrcomp. C'PE0.D'P=(c'P [(*273-01)] *273-11. F:.PE.=-:PeSer comp.D'P=PI'P:(gR).Re o. G'P=' R [*273-1. *263101] 4 -*273-2. 1-: W= XT {X = seqR'D'T 4 mins'TpQ'seqC'T D'T. D. BSPQ = A W. (RS)PQ C (A w*A)'fA. TRsPQ WA.' TRSPQ E (A jv*A)'A [*257125. *258242. (*27302-03-04. *259-02-03)] Here the temporary definitions of' 259 are revived. The second of the above inclusions might be changed] into an equality, but it is not necessary for our purposes to prove this. *273-21. F:Hp *2732..D'WAC C'R. tI' WAC C'S Dem. F. *25913.:) I: Hp. DI WA,4 = s'D" W"(A w*A)'A (1) F.*206-18. DI-: Hp. Xe DIW. D. D'X C C'R (2) F-.(1).(2). DlF:Hp.). Dc'T'ACGOR (3) Similarly I:Hp.). 'WA C C'S (4) F. (3). (4).) F. Prop *273211. F: Hp*273-2. TeU'TV.. Dlr\ D'DW'T== A [*206-2] *273-212. F: Hp *273-2.:. WAe Cls -- I.D r (A *A)'A e1 -.* 1 [*273-211. *259-141-171] *273-22. F: Hp *273-2. C'P - C'R. Pe connex. QC J.). WAe-+1.[I r (AW*A)'A e 1 -+ 1 Dem. F. *273-211-212-21. *2062. (*25903).) F:. Hp.):Te(A w*A)'An AW W.:). Te Cls-+ 1. D'TC C'P. seqR'D'TE D'T. 4 -[*272-2]). minD'TpQ'seqj'D'Trh e GI'T (1).(1).F:.Hp.): Te (A1*A)'A n PW. W.T tI'T n GIW'TT= A: F.259)14.173 12.We I -+ Cls. C1 r(Aw*A) e 1 — P 1 (2) F. (2). *273~2212.:) F Pr~op

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 *273-23. F: Hp*2732. P, Q eSer.C'P =O'R.C'Q

=G'S.TE (AW*A)'A.:). P ~D'T-=T;Q Dem. F.-*272151. *273-21..)F:Hp. T e CPW.) D.-P ~D'A wT =(Aw'T);Q (1) F.-(I). *259-16. D)F.Prop *273-24. F: T E(1S)pQ.:). D'T, PT eCis induct Dem. F.*120-251.:) F:. Hp.:): TeD'A~. D'TECls induct.:). D'Aw'T~eClsinduct: [*90-112]:):A(Aw)*T.:). D'TeClsinduct: [*273-2.(*27.3-03)]:): Te(IS)pQ.). D'T6Cls induct (1) Similarly F:. Hp.): T6(-RS)pQ.). G'TECls induct (2) F.(1).(2).)DF.Prop *273-25. F:P,QEq.CG'P= C'R.CG'Q=CG'S. TE(RS) pQ.:). D'TpQ =G'PK(I'TpQ =G'Q Dem. F. *273-1.)D F:Hp.D.P,QeSerncomp. C'P=D'P=dP"P.C'Q=D'Q=UI'Q (1) F. *273-1. *263-44.)DF: Hp.)D. ftP. fj!Q (2) F. (1). (2). *273-22-23-24. *272-43.) F. Prop *273-26. F:.P,Qeq.R,Seco.C'P=C'-R. C'Q='S.:): Te (IS)pQ. D). E!seq'D'T. E! in ins' TpQseqR'D'T Dem. F. *2 7 3 21. *263A47. *2 7 324. D F: Hp.- Te (RS)pQ. D. g! C']? A p-R"ccD'cT. [*250-122] D. E! seqRD'T (1) F. (1). *273-25.) F-: Hp.)D. g I TPQ'seqR'D'IT. [*250-121.*272-1] D. E! miln8'TpQ'seq.R'D'T (2) F. (1). (2).)DF. Prop *273-27. F:HP *273'2. Hlp*273-26.). (RS)pQ C P W.(RS)pQ CD'rA, [*273-26] *273-271. F: HP*273-26. TE (RS)pQ.). seqRD'TE D'TRspQ Dern. F.*273-2.:)F:-Hp. Hp*273-2.:). Te(RS) pQAiD'A w.).AW'TE(R$)PQ (1) F-.*273-2.:) F:Hp. Hp*273-2.Te (RS)pQ. E!Awl'T.). seq,'D,'T E D'Aw'T (2) F-. (1). (2). *273-27. D F: Hp. Hp *27-3-2. D). A ir'T c (RS) pQ. seqR'D'T E D'A w"'T. [*273'2.(*273-04)] D. seqR'D'T E D'TRspQ:)D F. Prop

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204 SERIES [PART V *273-272. I-: Hp *273-26..D"(~RS)pQ = R"GR Dern. I -. *206-401. F -: Hp. T E (RS)pQ. D'T = R'x. xo ECR.). x = seqR'D'T. [*204'71. *250-21]).D'IspQ'T = R'liX'x (1) F.*250-13.DF: Hp. D.DI'A R'B'R (2) F. (1).- (2). *90-131.:) I-:. Hp.): T (RspQ)*A. D. D,'Te R"G'CR: [(*273'03)] D D""(RS)pQ C R"G'C1R (3) F. (1).(*273-03).) F:. Hp.:): x e O'R. R'wx e D"(I(RS)pQ. D. R'R~'x e D"'(RS)pQ (4) F. (4). (5). *90-112. D F:. Hp. D xe (R1)*'B'R.). R'xE-D "(RS)pQ (6) F.-*263-43. *250-21.)DF: Hp.)D. C'R =COR1. B'R =B'iX (7) F. (6). (7). *263-141. *1 22-1141.) I-:. Hp.): xEU'CR.:). R' eD"(RS)pQ (8) F. (3). (8).:) F. Prop *273-28. F: Hp*272-26.:).TRSpQeI —+I.D'T.RspQ= C'P. P TRspQ;Q Dem. F.-*273-222.) F: Hp.). T~spQeI 1 F. *273-272.:)F: Hp.:). D'T~spQ = s'R"GcR [*263-22] =GC'R (2) F. *273-223. D F: Hp.).P D'TRspQ =TBspQ;Q. [(2)])P TRSQQ(3) F. (1). (2).(3). DF. Prop In order to prove TRspQ E P s-mor Q, it only remains to prove G'CTRQ = C'Q. *273-3. F:. Hp*27&2. TU,U(Aw*A),'A.:): D'TCD'IU.. TC-U Dem. F. *33-263.)F: TC-U.:). DTC D'U (1) F. *259-111.)F:. Hp.:): TCHT. v. UC T (2) F.-*33'263.)F: UC T. D'TC D'U.:). DT= DU (3) F.-(3). *273-212.)DF: Hp. UC T. DITC D'U.:).T= U (4) F. (1). (5).:) F. Prop

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SECTION F] RATIONAL SERIES 205 *273-31. 1-: Hp *273-26. TE(RS)pQ.yE C'S - U'IT. S'~y C PIT.. (ax, U). x =milne TpQ'fy. U E(RS)pQ. x =seqRD 11U Dem. F. *27 3-25. *250-121.) F: Hp.). (ax). x =minR'TPQ'y (1) F. *273-272.) F: Hp. x =minR'TpQ'y.:). (a U). U c(RS)pQ. D U = R'x. [*206-401]:).(aU).Uc (RS)pQ.,x= seq.RD U (2) F. (1). (2). F. Prop *273-32. F:Hp *273-31.X = MinR' TpQ'y. U c (RS) pQ. x = s~eqR'D' U.). xUpQy. TC-U Dent. F. *205K14.)DF:. Hp. ulx.):,(uTpQy): [*272-13]: (az):z eD'T: zPut. (T'zQy). v.uPz. (yQT'z) (1) F*272-2-42.) F:lHp. D. x 6D'T. (2) [*273-272] D. D'T C R'x (3) F. *273-272. D)F: Hp.)D. R'x-D'U (4) F. (3). (4). *273-3. D F: Hip.)D. T C U (5) F:.llp.,uRx.D: (az): zeD'T: zPu. r (zPx). v. uPz., (xPz) (6) F. *204-1.)DF:. Hp.): uPx. zPu. D. zPx: xPu. uPz.D. xPz (7) F.- (6). (7). (4). F:. Hp. uD eDU.): uPx. D. (az). z D,'T. 'uPz.,(xPz): xPu. D. (az)..ED 'T. zPu. r (zPx): [(2)]): uPx.D. (az). ze D'T. uPz. zPx: xPu.).(az).zecD'IT.zPu.xPz (8) F. *272-13. *273-23. D F: Hp. u c D' U. z e D'T. uPz. zPx. (U'u) Q (U'z). (T'fz) Qy. [(5)])(U'u) Qy (9) Similarly F: Hp. u cD'U. z cD'T.zPu.xPz.) yQ (U'u) (10) F. (8). (9).(10). F:. Hp.U ED'IU.D: uPx)(U'u) Qy: xPu D.).yQ(U'lu) (11) F.(5). (12)) F. Prop

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206 SERIES [PARBT V *273-33. F: Hp *27332.). y = min,' UpQ'X. X(RspQ' U)y Dem. I. *273-32.:) F: Hp. S'y C U' U. -*[*272-2A42]) Sty n UPQ'x = A (1) 4 -F. (1). *273-32.*205K14.)I-:Hp. D. y = mins' UPQ'X (2) F. (2). (*273-02). D F: Hp. D. x (RspQ' U) y: D F. Prop *273-34. I-: Hp*27331.).yEG'TRSPQ Dem. F. *273-3133. F: Hp.). (2U). UE (RS)pQ. YE WaRsQp'U. [*90l16.(*273-03)] D. (a3W). We (RS)pQ. yE'6(IY W. [(*273-04)]).ye U'TRSPQ: F. Prop *273-35. F: Hp *273826. D. IT spQ C'Q Dem..*273-34.) F: Hp. y E C'S. S'y C U'TBspQ.). YEU[TRSPQ (1) F. (1). *250-34. D F. Prop *273-36. F:Hp*273-26.). TRspQePs-i-h —r- Q [*273-28-35] *273-4. F: P, Qel. D. P srnor Q Dem. F. *273-11.D F: lp.). (SR, 5). 1, S E co. C'P = C'R. C'Q = C'S. [*273-36] D. (HR, S). TRSPQ E P s YQ: DF.Prop *27341. F: PEl. P smor Q. D. Qeq Dem.. *270-41. D F:Hp. D. Qe Ser n comp (1) F. *151-18. *123-321. D F: Hp. D. C'E N0, (2) F. *115 1L DF: Hp. D.D'Q = (U'Q (3) F. (1). (2). (3). *2731. D F. Prop *273-42. F: P E q. D. = Nr'P [*2773'4-41] *273-43. F. lE NR [*273-42.-*2.56-54] The following propositions are easy to prove: F: Q E Ser ^ C"N. P E r.). Q x P e whence:ae NR riCl'Ser. C"a = N. D. a l=l; and F: P E q. Q e Ser n C"No. X E CGP.). x 4,Q e Nr'Q n Rl'(Q x P). Q x P E 17, whence, from the fact that all q's are similar, F: P E. Q E Ser r CG"K0.). D Nr'Q n RI'P. Tbus an iq contains series of all the order-types composed of K0 terms.

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*274. ON SERIES OF FINITE SUB-CLASSES OF A SERIES. Summary of *274. In the present number, we shall be concerned with the construction of a rational series consisting of the finite existent sub-classes of a progression. When the finite sub-classes of a progression (excluding A) are arranged by the principle of first differences, the result is a rational series. When they are arranged by the principle of last differences, the result is a progression. These two propositions, with the consequent existence-theorems, are to be proved in the present number. We define "P," as P1i with its field limited to finite existent classes. (For the definition of Pd1, see *170'01.) In the present number, we shall be chiefly concerned with P, when Peo, but it has interesting properties in many other cases. Our definition is P, = PC1 t (Cls induct - t'A) Df We shall be concerned in this number not only with P,, but also with PIlt (Clsinduct-t'A). This is Cnv'(P),. Thus if we put P = Q, the hypothesis that Pe E as used in studying PIC (Cls induct - iA) is equivalent to the hypothesis that Q e as used in studying Cnv'Q,, i.e. Q,. Thus the study of Pc1 and P1I with their fields limited to inductive existent classes may be replaced by the study of P, in the two cases where (1) P e f, (2) P e f. The second case is the simpler, and is considered first. We have first, however, a collection of propositions which only assume that P is a series. Since an inductive existent class in a series must have a maximum and a minimum, we have *274-12. F:: PeSer.:.aP,3. =-: a, / e Cl inductC'P - 'A: (az). z e a -/3. a n P'z = / n P'z We have *274'17.: C'P E e 1. D. C'P, = C1 induct'C'P- tA

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 Whenever P is a series, P, is a series (*274'18). If P has a last term, the class consisting of this last term only is the last term of P,,; if P has no last term, P, has no last term (*274'191). If C'P is an inductive existent class, the first term of Pn is C'P (*274'194); if not, P, has no first term (*274-195). Hence if P has no last term, P, has no first or last term, and we have D'P = ('P, (*274-196). Thus of the characteristics used in defining I, we have P, e Ser whenever P e Ser, and D'P, = Q'P, whenever E! B'P. We next prove *274-22. F: Pe. D. PE, which, in virtue of what was said above, is equivalent to P e 2. D. P1,c (Cls induct - t'A) e 1, that is: The principle of last differences applied to the inductive existent sub-classes of any well-ordered series gives a well-ordered series. To prove *274'22, since we already know that P, is a series, we only have to prove that every existent sub-class of C'P, has a maximum with respect to P,. This is proved as follows. Let K be any existent sub-class of Cl induct'CP - t'A. Consider the minima of all the members of K: these minima all exist, because K is composed of inductive classes. Then in virtue of the nature of the principle of first differences, members of K which have a later minimum come later than those that have an earlier minimum. Hence if we consider minp'C, the classes whose minimum is the maximum of minp"c (which exists, because P e 2) are later than any other members of K. Put x1 = maxpsminpKcg. /c K= KF n minpxl. Thus Kc consists of those members of K which have the largest minimum, and members of K1 come later than any other members of K. Similarly the latest members of K1 will be those that have the greatest second term. That is, if we take away the (common) first term from each member of K1, and if XI is the resulting class of classes, we have to apply to XI precisely the same process as we have already applied to K. Thus we are led to put xa = maxp minp"c. K1 = K n minp'xi. X = (- t'l)"X1, X2 = maxp'minp"X1. K2= C n minp'x2. X = (- 2)cK2, and so on. The series x,, xe,... is an ascending series in P, and is therefore finite, by *261'33. It therefore has a last term, say x,. Then the class lxC, t tx2 v... V LtxY is a member of K, and is easily shown to be its maximum. Hence every existent sub-class K of C'P, has a maximum, and therefore P, e 1.

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SECTION F] ON SERIES OF FINITE SUB-CLASSES OF A SERIES 209 In order to symbolize the above process, we put Pnlc = maxp'minp"c Dft, v 4 -Tp'ic = (-t'Pm'K)"(tK n minp'Pm'K)- t'A Dft, MpeK = Pm"(Tp)*.'K Dft. v 4 ---Then Pm'K is what we called xI, Tp'K is what we called X,, (Tp)*,' is the class K, X1, X,... X-,, and Mp'K is the class x1, x2, x3,... x,. Thus what we have to prove is Mp'K = max (P,)'c, which is proved in *274'215. We prove next *274-25. FI: P e.. P, e For this purpose we use *263'44, namely o = n - t'A n P (a'Pi = a'P. E! B'P). Thus it only remains to prove D'(P,), = D'P,.* E! B'P,. E! B'P, follows from *274'195, and D'(P,) 1 = D'P, is proved without any difficulty; hence our proposition follows. From *274'25'17, by substituting P for P, we obtain *274-26. F: P e o. D. P1c I (Cls induct - A) e w. C'PTic (Cls induct - 'A) = Cl induct'C'P - t'A whence it follows immediately that *274'27. F: a e oK. D. C1 induct'a e. Cl induct'a- t'A e Ro I.e. a class of Ko terms contains Ko inductive sub-classes. We now have to prove *274-33. F: P e.. P, e In virtue of *274-17'27, we have C'P, e K,; and by *274'18, P, e Ser. Thus it only remains to prove P, e comp. D'P,= ('PP,. The second of these results immediately from *274'196. As for P, e comp, if aP,,3, a v /3 e Cls induct, and therefore a! p'P"(a u v3); but if x e p'P"(a v y/), we have aP, (/ u I'x). (/3 u Lx) P,Y; hence P, C P,2. This completes the proof that P, e q. The proposition holds not only if P e o, but if P is any series which has no last term and whose field has R, terms (*274'32). Finally, we deal with the existence of 77 (*274'4 — 46). If P e co, P is similar to Pic (Clsinduct- t'A), by *274'26; and if T is a correlator of R. & w. III. 14

210 SERIES [PART V these two, TP,1 is an q whose field is C'P (*274-4). Hence the existence of i7 in any type is equivalent to the existence of c in that type (*274-41). Hence we have merely to apply previous propositions on the existence of o. *274-01. P,1 = Pe, (Cis induct - L'A) Df *274-02. PmlK = maxp'minp "K Dft [*274] *274-03. Tp'K = (t 'Pm'"K)""(Kn A ilinp'Pmi)K- I"A Dft [*271 4] 4 --- *274-04. MpIc= Pm"(Tp)*'K Dft [*274] *274-1. F: aP,3. _.a, E Cl induct'CIP - /A.A a - P"(/ - -a) [*170'I. (*274-01)] *274-11. F: P e Ser. ae C1 induct'CP - iA. E E Inin'a. E maxp'a [*261P26] *274-111. F: Pe Ser.e BTE!B'P. acClinduct'CGP..H p'P"a Dern. -. *274-11. I -: Hp. a!a.).maxp'ae D'PI. [*c205-65] 2. 1! 3PIor (a -. (1). *40-2.)-.Prop *274-12. F::PeSer.D:.aP,1/.P: a, /3 E CI induct'CIP - t/A: (az). z e a - /. a A P'z = 8 A P'z Dem. F-.*170 2. F~:.at, 8f Clinduct'CcPP-t"A:((z).zZ ar -P.a an P~z= o3 Ylz:.aP,,8 (1) F.*2'74~11.)I~: Hp.aP,1/3.)~. E minp'faOI- - P"(/3-a)}. [*170-23.*205r192]). (az). a A PIz= n PA (2) F()(2).) F.Prop *274-13. F-. PI,, (Cls induct - t/A)= Cnv'(P), [*170101. (*27401)] *274-14. I-:: PESer.):. aPl,, ~ (Cls induct- t'A)}/3 4- - a, / e Cl induct'C'P - t'A: (az). z e / - a. a A P'z = /3 A P'z [*274-12-13] *274-15. I-:a,/3 e C1 induct'C'P - ViA./3 C a./3a.)a. aP,1/ [*170-16. *274-1] *274-151. F: a c Cl induct'C'P - 1. x e a.). aP,1(t/x) [*274-15]

SECTION F] ON SERIES OF FINITE SUB-CLASSES OF A SERIES21 211 *274416. F: jP, C.GP'-,,e 0 v Dem. F.*274-151.:)F:CG'Pr-eOvI.:).f!P, (2) [*274-1]).P,, = (3) F-. (1).(2). (3).)DF.Prop *274417. -: C'P —E,1-.).C'P,= Clinduct'C'P- t'A Dem. FI-.*274-151.. F. Cl inductCP - t'A- 1 C DIP, (1) F. *274'151.)DF: xeG'CP. C'P +tfx.). txe 1PP, (2) F.(4). *2 741.DIF. Prop *2744171. F: P2 C. J. XPy.. (t'a) P, (t'y) [*274-1] *274418. F:Pe Ser.)D. Pn, e Ser Dem. F. *201-14.)D F-:. Hp. z e a - /3. wE /3 - ry. a A 3n P 'ZPz.8 /3, P'w ey en ~ zPw.).za-,y-cAPl'Z=yrP'Z (1) F. *201-14.DF:. Hp (1).)D:wPz.)D.we a — y.a A Plw= y rP'w (2) F. (1). (2). *202-103. *274-12. D)F:Hp. aP./3./3P,,ry.D. aP~y (3) F. *274 11.)D F: Hp. a,/3, ECL induct'C'P - t'A. at /3. D. (az). z = minp' t(a - /3) v(/38-a)j. -4 -4 [*205-14] D). (az). z e(a -/3) v (/- a). a nPIZ-/AP'Z. F.(3). (4). *170-17.) F. Prop *274419. F: Peconnex.PC-J.:). B'P=t"'B'P Dem. F. *274-151.)F. Cl induct''P - 1 C D'PI (1) -4), %. -4+ %J F*202-524. 14-2

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212 SERIES [PART V F-. (4).)D -+%.,%. F: Hp. xe B'P.). i),3). Cl inductCP'P- t'A. H! t'X-/3-P"'(18- L'X). [*274-1]:). tl,, ',(5) F. (5). *274-17.:)I-:llp.:). 'B'P C B'(P (6) F.(3). (6) F.)I-Prop *274-191. F:. P econnex. P2 C.J.): E! B'P.D.BRIP, = L'B'P: *274-192. F.Pe connex.P2 CJ.D:E! BrP. -. E! B'Pj [*274-191] *274-193. F.B'P, =t'O'P n(Cis induct -t'-1) Dem. F.*274K15-1.:)F:C'PeClsinduct-tfA-I.:).C'PeB'P,, (1) F.*274-16-17.:)F:C'P~E(Clsinduct-t'A-1).:).C'P-eC'Pq, (2) F. *274-15.)F: aeClinduct'C'P-t'A.xeC'P-a.).(avt'lx)P,,a (3) F. (3).)F. Cl induct'G'P - tA- t'IC'P C U'P?7 (4) F.(4). Transp. *274-1.F. B'P,, C (C induct'C'P - t'A) n tCP(5) F. (5). *274-16. PF. B'P17 C (Cls induct - t'A - 1) A CC (6) F. (1). (2).(6).) F. Prop *274-194. F: C'PeClsinduct-tfA-1.). B'P, =G'P [*274-193] *274-195. F:C'P, ECls induct.). I3'P, = A [*274-193] *2744196. F:PeSer.,-.E!B'P.).D'P,,=G'P', De. F. *274-192. D)F: Hp.D. B'P = A (1) F. *274-195-16. *261 24.:) F: Hp.:)..B'P =A (2) F.(1).(2).) F. Prop The following propositions give the proof of P C Sil.:). p E fi2 (*274-22). *274-2. F:Pe fl.lc CG'P, aK. D.E!Pm'IcK. Prn,',ceminp"K [*274-1'11. *250-121. (*274-02)] *274-201. F:fe Tp'CK. ~(aa). a EK. minp'c = Pm'xI./3= a- t'Pm,'Kai3 [(*274-03)] *274-202. F: E! Pm,,'Ic. D. E! TP' Cc[(*274-03). *14-21] *274-203. F:. Hp *274-2.>: Tp'ic-=A. E. K A Minp'Pm'x"/ = t't'Prn'K Dem. F. *274-2202.:) F:: Hp.):. T p'K= A.u:rs3f)aE K. fiinllllp Pm,'K., -~ OPt'P'K! /3: 4 -[*13.191] BaECKAn Minp'Pm'K. c.t'Pm'K=cA: 47 -[*274'2] a: C EK AN minlpPm'K a=c~ t 'Pm'K:::) F. Prop

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SECTION F] ON SERIES OF FINITE SUB-CLASSES OF A SERIES21 213 *274-204. F:K C C'P,,. K (Tp)* X.. XCCC'P,7 Dem. I.*1 20-481.*274-201.) K C CIS induct. E! TPK.:).' TKC Cls induct (1) H.-*274-201.H: K C C1C'P. E! Tp'K. D-TCK C C1C'P - tA (2) H.(1). (2). *274-16.) H K C C'P.E!Tp'CK.)Tp'K CCC'P (3) F.(3).JInduct.:)IF. Prop *274-205. H:PE Ser. E!Pm' Tpx.).(Pm'X) P(Pm'Tp'X) Dern. H.*274-2O1. *205'J'21.)H:Hp./3e pX.I3PP X () H.- *205115-)II. (*274'02). D H: Hp.)D. Pm.'TP'-

ke s'Tp' X (2) F. (1). (2).:) HF. Prop *274'206. H: Hp *274-205. K (Tp)*X.). (Pn'CK) P (PM' TPX Dem. F. *14-21. (*274-02).DH: E! Pm,'Tpx. D. E! Pm'x (1) F. (1). Induct. DH: Hp.D. E! P,1,K (2) H. (2). *274-205. Induct. D H. Prop *274-207. H: P e fl. K (Tp)*X. Pmc X =maxp'Mp'K.) ~E! P~'TP'-X. TP'x= A Dem. I-.*274-205. Transp. D H: Hp. D. ~E! PM' TPX [*274-204-2.TraDSP]).TPCX = A: D H. Prop *274-208. FH:. P e UcC C'P7 K. 4-.-4.' A e(Tp)*'K: (ax). K (Tp)*X. X ri mfilp'Pm',X =t't'Pm,' X. Tp'X =A Dem. FH.*250-121FH: H p. D.E!maxp'Mp'K (1) HF. (1). *274-207-203-204.:) H. Prop *274-21. H: 3e Tp'K.). 3V t'Pm'K eK [*274-201] *274-211. H:K(Tp)*x./3eX.).f3vPm"Tp(K ~ —X) eK Dem. ye p'X.)y.Y.yv Pm,"Tp(KHJTp'X)E K (1) FH.*274-21. (1). Induct.: FH. Prop

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214 SERIES [PART V *274-212. F:P efl.KiC C'P~,! K.).Mp'KEK Dem. F. *274-208-211.:) F: Hp.:). (ax). I(Tp)*X. Tp'x= A. L 'Pm`X eX. 'lPm'X vPm",lTp(KHX) C K. D F-. Prop *274-213. F:PeSer./KCG'P~. aCK c.K(Tp)*X.P'Pm'XAMp'KCa.). Dem. a - ~~~~~~~ (P'Pm'X A Mp'K) e X FI-. *274-201.) F: Hp. K = X. D. a - (P'Pm'X A MP'C) =a. [*13-12] D.a - (P'P,,'X AMp'K)ef (1) F. *274-206.)D F-: Hp:8 -9 K. P'PM'X Ai Mp'K C,&.:).8 - (P'P,,,'X A Mp'K) e /3 fcK. P,'Pm' Tp'X A Mp'K C,8.:). P"Prn'X A Mp'K C,3. Pin' XC 3,8- (P'Pm'lX A Mp'K)J E X. Pmc X C {/3 - (P'Pm`X A Mp'ic)1. [*274-201]:). t8- (P'Pm'X AMp'Kc) -t'Pm,'X} CTp'x. [*274-206] D. {3-(PrTp' A CTK) TpX (2) F.(I).(2). Induct.:)F. Prop *274-214. F:Pef2.KCC'Pq.a6CK-. t'MP'CK.:).aP),(MP'K) Dern. F. *274-212. Fl:. Hp.):Mp'ce Cls induct: (1) [*170-16]): Mp'K C a. 2D. a P,(Mp'Kc) (2) F. *274-11. (1).:) F: Hp. a! Mp'Kc - a.:). E! miup'l(Mp'Kc - a). [*205-14.(*274-04)] D. (1X). Kc (Tp)* X. Pm'`X C, a. P'Pm'X A Mp'CK C a. [*274-213] D. (HX). Kc (Tp)* X. Pm'Xr Ca. a - (P'Pm'X Al MP'C) e X. P'PM' X A MP,'I C a.[*274-201] 2 ~,z). Kc (Tp)* X. z= nliinp'{Ia - (P'~Pm'X AN Mp'K)J zP (PM,'X). PT'Xm \ Mp'K C a.[*31-18] D. (az). Z ea -MP'K.-MP1KAnP'z C a. F.(2). (3).)DF.Prop *274-215. F: PC&.1. C C'P,. K.:). Mp',crniax (P,)'K [*274-212-214]

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SECTION F] ON SERIES OF FINITE SUB-CLASSES OF A SERIES21 21.9" *274-22. F:P e Q P.). Dem. F. *274-215.) I- lip. D. E!1! max (P,)"Cl ex'C'P~. 11*250-125] PI. P I ei:F.Prop The following propositions constitute the proof of PeEco.:)~Pnew (*274-25). *274-221. I-:P ESer. P'maxp'la ECisinduct.a eCl induct'C'P -t -,'1'P. /3= (a - t'rnmaxp'la) v P'rnaxp'a a aPlf Dern. F. *205-55.)F: Hp.BRIP ea.:).[! a - tmaxpla (1) F. *202-511. DF: Hp. B'P,ea.D. B'Pe P'maxp'a (2) F*93-101. D F: Hp.,, E!B'P.:).:! P'maxp'a (3) (4) F.*120-48P7l.:)F:Hp.:)./3,EClsinduct (5) F. *205-21. *200-361.:) F liHp.:). /3 ri P'niaxp'a - a A P'maxp'a (6) F. (4).(5).(6). D F: lip. D. a, /3 Cl indu'ct'C'P -t'A. raxp'a e a-/. a A P'maxpla = /3 n P 'maxp'a. [*274-12] D. a P,,/: D F. Prop *274-222. F: Hp*274-221.aP,,y. maxp'aevy.). /P,,,y Dem. F. *274-12.)F:Hlp.).(a[z).z Ea -ry. z 4maxp'a. ar P'z = y AP'z. [*201-14.*205-2l. Hp]:). (Hz). z e /-ry. 3 A PIZ =, e7 API'Z. [*274-12]:). /3P,,ry::) F. Prop *274-

223. F: Hp*274-221..aP,)y. rnaxp'ar-.JEfy.ry#/3.)./3Pty Dem. F. *274-12.:)F:. Hp.:): (az).z ea-y- t'lmaxp'a. a AP'z=ry AP'z.v. a A P'maxp'a =y A P'maxp'a (1) F. *201-14. *205-21. F:.Hp:(Hz).ze a -y - traxpla.a nP'z -y %P'z:)./3Pny (2) F.*205-21.:)F:Hp.anP'maxp'a=ryAP'maxp'a.:). a - tfrnaxp'a =7 Ay P'rnaxp'la (3) F.*20 2 101.)F:Hp. yC P'm axpla vP' maxpla (4) F.(3).(4).)F:.Hp.anP'maxp'a=, ynP'maxp'a.):,yC/3: [*170-16.(*274-01)]):ry +/3.)./3P,1y (5) F.(1).(2). (5). D F. Prop

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216 SERIES [PART V *274-224. F:Hlp *274-22l. tP,,y.,94 y.. /3P117 [*274'222-223] *274-23. 1- Hp *274-221). a(P,),O/ [*274-221P224. *204-72] *274-25. F: Peo.).PqEO, Dern?. F. *274-191-17. F. *263-412.*274-11. 1-: Hp. a e CI induct'G'P - t6A.). P'rnaxp'la E Cis induct (3) F (1). (4). *274-195. *1 21-323. [*263'44] D. P-n D F F.Prop, *274-26. F:PEWo.).Pl,~(Clsiinduct-t'A)ew. Dem. CII (Cis induct - t'A) = Cl inductP'CP -tA F.*274-25.:)F:Peco.Q=P)Qq ewc (2) F.*274-17.) F:P eco. Q =P.).'Q, =Cl induct'G'P - t'A (13) F () ().(3.).Prop *274-27. F a Eo.). Cl induct'a E,. Cl indluct'a -t'A eN Dem. F.*263-101.)F:Hp.).(g[P).P~ew. a=C'P. [*274-26] D. (2jM).- MEWo. Cl induct'a - tI C'M. [*263-101] D. Cl induct'a - t'Ae E, (1) [*123A4] D. Cl induct'a e No(2) F.(I). (2). FProp The following propositions constitute the proof of PE6co D.).PqEfl (*274-33). *274-3. F: Pe Ser. aPF4l.wep'P"P1(a Wfl).D. aP~,(/3v t'x).(fl v tfx) PS3 Dem. F. *200-53. DF:Hp.zea. D. Pz(vlxA'z(1) F. *200-5) F:Hp.zea-/3.D.zea-(/3vt'x) (2) F.-(I).(2). *27 4'12.DF: lHp.).aP, 6 v t'x) (3) F.*200-5.*170-16. DF:Hp.D.(/3vt'x)P,,fl (4) F. (3). (4).:) F. Prop

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SECTION F] ON SERIES OF FINITE SUB-CLASSES OF A SERIES 217 *274-31. 1: Pe Ser.- E!EB'P.). P e Ser n comp Dern. F. *274-1. *120-71. D H: aP,q/. D.a v/e, s Cls induct - t'A (1) F. (1) *274-1. D: Hp. arP,,8. D. E! m~axpl(Ca v 8) [*93-103] D. H! P'maxp'l(a v 9). 4 -[*205-67] H i!p CP6 (a v p8). [*274'3]).aP,2, (2) F.(2). c*274-18.:) F. Prop *274-32. F: Pe Ser n C"IIM. eE! B'P.) 7 Dem. F.k*274-31. F: Hp. PPESer n comp (1) F.*274-196. DF:Hp.D.D'P,,=PP, (2) F.- *274'27-17. D F: Hp.D. G'P, e N0 (3) F. (1). (2). (3). *273-1. D F. Prop *274-33. F: Pew.).P El [*274-32.*263-101P11-22] This is the principal proposition of the present number. 4 -*274-34. F: a e No.).D n C'(Cl induct'a - t'A) Dem. F. *263-101.) F: Hp. D. (HP). P E C G'P - a. [*27433-17] D. (2[M). Mefl]. C'M= Cl induct'a - tIA: F F. Prop The following propositions are concerned with the existence-theorem for 77. They all follow from *274-33. *274-4. F: P c w. T= ffP sinor IP1,, (Cls induct - t'A)}. D. T;P, e n C'C'P Dem. F. *274-26-17. D F: Hp.D. PIT= CG'P, (1) F. (1). *151-11 131.) DF: Hp.)D. T;P, smorP,. C'T;P, = C'P. [*274,33.*273-41]:). T;P?,C. C'T;P- = CP)F. Prop *27441. F:a!o t'P E. H!'n".t'P Dem. F.*274-4.)F:Q cnt'P.). (3R).Refl.C'R=C'Q. [*64-24] D.3b! 1 A t'P (1) F. *~273-11. D: R,6 -q n tIP.. (HQ (~). Q E w - CIQ = C'R [*64-24] R.a! ovt'P (2) F. (1). (2). D)F. Prop 4 -*274-42.

F: Nae.:). g!I n CMa [*2744-26. *263-17. *250-6. *263-101] *274-43. F No = C"rn [*273'1. *274-42] *274-44. F: a!No n ta = a i [7 n to'la [*263-131. *27441] *274-45. F: No (!(x). a H!',q t"'lx [*263-13.*274-41] *274-46. F: Infin ax (x). a"i n t33'X [*263-132. *274-41]

*275. CONTINUOUS SERIES. Summary of *275. The definition of continuity to be given in this number is due to Cantor. A different and not equivalent definition was given by Dedekind: series which are continuous in Cantor's sense are also continuous in Dedekind's sense, but not vice versa. Cantor's definition has the advantage (among others) that two series which are continuous in his sense are ordinally similar, which is not necessarily the case with series that are continuous in Dedekind's sense. Dedekind's definition of "continuous series" is, in our language, "series which are compact and Dedekindian." Cantor's definition (after a certain amount of simplification) is "series which are Dedekindian and contain an K0 as a median class." In the case of the real numbers, the rationals are a median class of this sort. An equivalent definition to the above is that a continuous series is a Dedekindian series whose converse domain is the derivative of a contained rational series (*275*13). Following Cantor, we shall use 0 for the class of continuous series. In what follows, we prove first that the series of segments of a rational series is a continuous series, i.e. *275'21. h: P.) *. 'P 9 The contained No is P"C'P. The proposition follows at once from *271*31. On its importance, see remarks on *275'21 below. From this proposition, it follows that if Vr exists in any type, 0 exists in the next type (275'22), whence the existence of 0 in sufficiently high types follows from the axiom of infinity (*275'25). To prove that any two continuous series are similar, we use *271'39. By the definition, if P and Q are continuous, they contain respectively two median classes a and 9, such that P t a and Q, / are rational series. Hence by *273'4, P asmor Q /, and therefore PsmorQ, by *271'39. Also obviously P e 0. P smor Q. D. Q e 0. Hence *275 32. F: P e. 0. 0 ==Nr'P and *275'33. r-. e NR

SECTION F] SECTION F] ~CONTINUOUS SERIES29 219 *275-01. 9 = Ser n Ded n rned"No Df *275-1. HPe0.E.PeSer nDed. a! No nmed'P [(*275-01)] *275-11. F:. Pe9.=-:PeSerADed:(ata).aEN,0.3p'a=U'"P.aCC'P [*275-1. *271P2] *275-12. F:: P eO. =:. PeSer ^Ded:.(Ha): a e o: XPY. D~,Y a! aA P(x -y): aC CTP [*275-1. *271'1] *275-13. I-:. Pe9.:P eSer rADed: (HR). R CP.-Refl.p'C'RG1P Dem. F.*273-1.*271-2.) I- P e Ser rn Ded. R C P. R q ~. 8p'C'R = PP.). C'1E No -. C'] e med'P. [*275-1] D.Pe9 (1) F *271P16. DF:camed P.O =a n D'P nCAPP.).IOmed P. (2) [*271-15] D. P j3EcCOMP (3) F.*1 23-17.DF: Hp (2). Pe Ser. ae No iC1'C'P.)./e o ri CL'C'P (4) F.*271-1. DF:j8medP.:).P"/3,=D'P.P"3= 'P() F.-(5). *3 7 41. (2).:)F Hp (2).).D'(P j3) =fl. ((P/3) =/3 (6) F.(2). *271-2.) F: Hp (4).). 8pl'C6(P ~3) = WP (8) F.(l).-(9).:) F. Prop *275-14. F. 9= Cnv"9(Dem. F. *214-14. *271-11. F::

PcSer ADed. ac No imed'P.. P cSer nDed. a c Nr ^medlP (1 F. (1). *275-1.) F. Prop *275-2. F: Pen. D. g'PEc Serri Ded. P,"C'P cK0. P"IC'P emred'ls'P Dem. F. *214-33. D F: Hp. D.sPeSernDed (1) F. *204-35. F:Hp)P"IC'P sm C'P. [*273-1. *123-321]) P"C'Pe 6 (2) F.*271-31. *273-1. F: Hp).P"C'VP emed'StP (3) F.(1). (2). (3).) F.Prop

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220 SERIES [PART V *275'21.: P e...s'P e [*275-2'1] This proposition is of great importance, particularly in the theory of real numbers. We shall define the real numbers as segments of the series of rational numbers, in order to be sure of their existence. Thus if P is the series of rational numbers, S'P, which may be taken to be the series of real numbers, is continuous. If P is the series of rational proper fractions, excluding 0, s'P is the series of real proper fractions together with 0 and 1: this series is continuous in virtue of the above proposition. The above proposition is also useful as enabling us to deduce the existence of 0 from that of V, and thence from that of o0, and thence from the axiom of infinity. A rise of type is, however, required for the existence-theorems, which are given in the following propositions. *275*22. F:! 7 too'. D.! O r t1"a Dem.. *64-55.:) F:! ' n to'a.). (HP). P e q. C'P C t,'a. [*63-371]. (gP). P e v. C'P e t'a. [*275'21]. (Q). Q e 0. C'Q C t'a. [*64-57] D. a! 0 n t"'x:) F. Prop *275-23. F:! Ko n t'a. D. g! n tl1'a[[*274-44. *275'22] *275'24. F: H! No (x).:).! 8 tn'x [*275-23. *6431-312. (*65-02)] *275'25.: Infin ax (x). D. [! 60 t4'x Dem. F. *123'37. D: Hp. D. H! K0 (t2'x). [*275'24] 3.! 0 n t22t2zx. [*64'312] D. [! ^ t4x: D F. Prop *275'3. F:P, Q e 9.. P smor Q Dem. F. *275-13. D h:. Hp.: P, Q e Ser n Ded: (IRS)R, Seq.R P.. P. S. C'Remed'P. C'Semed'Q: [*204-41]): P,Q eSer Ded: (a, / 3).amedP. medQ. P a, Q 8e q: [*273'4]: P, Q e Ser n Ded: (ac, f). a med P.3 med Q.(P a) smor (Q Bf): [*271-39] D: Psmor Q:.:). Prop *275-31. H: Pe. Psmror Q. 3. Qe0 Dem.. *271-4.): P smor Q.! Ko n med'P..! o med'Q (1) F. 204-21. *214'6. F-: P e Ser n Ded. Psmor Q. 3. Q e Ser n Ded (2). (1). (2). *275-1.) F. Prop *275'32.: P e 8. D. = Nr'P [*275-3'31] *275'33. F. 0 e NR [*275-32. *256-54]

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*276. ON SERIES OF INFINITE SUB-CLASSES OF A SERIES. Summary of *276. The subject of the present number bears the same relation to 0 as that of *274 bears to r. We shall consider, in the present number, the arrangement of all the infinite sub-classes of a series (together with A) by the principle of first differences, i.e. the relation PC1 C (- Cls induct v L'A), where P is the given series. This relation we will call Pe. It consists of PC1 with its field limited to terms not belonging to C'P, (*276'12). It will (under a certain hypothesis) contain a part similar to P,, namely P,, with its field limited to complements of finite sub-classes of C'P. Hence if P e o, Pe will contain an I, whose field is composed of the complements of members of C'P, (*276-2). The field of this Vq will be a median class of P0. We shall find, also, that P0 e Ser, if Pe 12 (*276-14), and P e Ded, if

Pe Q infin (*276-4). Hence *276 41. F: Pe. D. Poe 8 Also, since Peco.3.CI'C'GPe 2, and since C'P,e0o, we shall have C'Poe2K~ (*276'42). This result is important, since it gives the proposition *276-43. F. C"0 = 2N The proof that Pe is Dedekindian if P is an infinite well-ordered series is somewhat complicated. We proceed by, proving that every sub-class of CG'P has a lower limit or a minimum. In this proof, we observe first of all that C'P = B'Po. A = B'Po (*276-121). Hence C'P is the lower limit of the null-class, and A is the minimum of t'A; also if K is any existent sub-class of C'Po, other than t'A, we have limin (Pe)'K = limin (Pe)'(K - t'A). Hence if we can prove KC C 'Pq. b! K. A- K.). E! limin (Po)'K (A), we shall have C1 ex'C'P C ('limin (Pe),

222 SERIES [PART V whence, by *214'12'14, we shall have P e Ded. Thus we have to prove (A), i.e. c CD'PO.! Kc.:). E!limin(Po)'K, which is *276'39. To prove this proposition, consider minp'(s'K-p'c). This exists unless K el; it is the first term which belongs to some members of K but not to others. Those members of K to which it belongs precede (in the order Pe) those to which it does not belong. Let us call those to which it belongs Tp'K, so that ^ 4 -Tp = X I[X = IC n e'minp'(s'K - p'c)}. Put also Pm'K = minp'(s'c- p'c) Dft, v 4 -so that we may put Tp'c = Kc n E 'Pm'K Dft. Then if we put A = K X (X C c. X + K<), Tp and A fulfil the hypotheses of *258, and we have A (Tp, K). The series A (TP, c) proceeds to smaller and smaller sub-classes of K, of which any one, say X, consists of terms which come earlier (in the order P.) than any other sub-class of K not belonging to X. By *258'231, the series A(TP, K) has an end, namely p'(Tp*A)Kc. If this is not null, it must consist of a single term, which will be the minimum of K (*276'33). But if it is null, we proceed as follows. Put Ptl'K = s'/ {(X\). X e (Tp*A)K. 7 = p'X n P'Pm'X} Dft. Then Ptlc' will be the lower limit of c. In the first place, we easily prove that, since p'(Tp*A)'c = A, if X e (TpA)'K - t'A, Pm'X and Tp'X both exist (*276'341). Hence every member of K has predecessors in Kc, and K has no minimum. In the second place, we show that X {A (Ti, K)}]/. gj! /s. D. (Pm'X) P (Pm'FX) (*276-34-342), and that a e X.). p'X n P'Pm'X =a n P'PmX (*276'353). Hence we find that X tA (Tp, K)},ti. a. e. p'X n P'Pm'X = p'L n P'Pm'X =a n P'Pm,'.). pX n P'Pm'X C p'L n PPmL. (p'l n PAP, n't) A P'PmCX = p X n P'P,, whence it follows that x e (Tp*A)'Kc - t'A. p'X P'Pm'X = Ptl'K n P'Pm'X, whence, by what was stated above, e (Tp*A) a e X D. a n P- = Pt n PX (*276354). Xe (Tp. A)'K. a eX.:). a A P'P,m'X = Pti'K A P'Pm'X (*276'354).

SECTION F] ON SERIES OF INFINITE SUB-CLASSES OF A SERIES22 223 Again, if a E6 K, the product of all the members of (Tp*A)'ic to which a belongs is a member of (Tp*A)'Kc to which a belongs, but if we call this product X, Pm'"X —.'e a (because, if Pm',,X e a, a 6 Tp"'X, which is contrary to the definition of X). Hence we have a 6 K.).(Pj1',c) P0 a (*276-36). It only remains to prove (PtI'c)

P/3.). (aa).a K.aP0/38 (*276-37). By the hypothesis, and the definition of Ptj'K1, we have -+ -+ -+ (az, X). XE (Tp*A)'K. ZE ~pX A P'Pm'X - /3. Ptj'"c A P'1z = /3 r P'z. Since this involves E! Pm'1X, it involves X +=A, hence, by what was stated above, it involves (a1z, X, a). X c (Tp*A)'K. a 6 X. Z z6 a A% P'Pm',X - /3. Pt1'KC A P'z =8 /3 P'Z. Hence we obtain /3 A P'z C Pti'K Ar P'1Pm'XX and Pt1'K A P'Pm'"X = a A P,'Pm'X,, whence /3 A P'1z C a. Hence, by *170-11, we have aP9/3. This completes the proof Of Ptj'K = tl (P0)'K (*276-38). Hence, combining, the two cases, we find that K has a minimum if a1! p'(Tp*A)'K1, and a lower limit if e'-'r2 [! p'1(Tp*A)'K1. Hence E! lirnin (P0)'K, in either case (*276-39). This completes the proof of Po e Ded if P e f infin. *276D01. Pq P,1 (Cls induct v i/1A) Df *276-02. A=&/3(/3C a./3ta) Dft [*276] *276-03. Pm'X = minp'(s'X - p'CX) Dft [*276] A +~*276-04. TP= X!{/A X EPmX Dft [*276] *276-05. Pti' = 't). Xe(Tp*A)'K -ti/A. rY=p'X iP,'P,'X}I Dft [*276] *276-1. I- aPe/3.. a, /3 e (Cl'C1'P- Cls induct) v i/A. a! a - / -P(/ a) [*170-1. (*276-01)] *276-11. I-:: Pe f2.):. aP03 Ea,/ e (CII'C'P -Cls induct) v A: (az). z a -/3 a A P'1z =8 /3^ P'1z [*251P35. (*276{01.)] *276-12. -: C'Prj,61.).Pq=Pci~(G'P,) [*274-17.*276-1.*170-1] *276-121. I-: C'P '-.' eCls induct.. B'Pa = A. B']P9 =' ci- C'Pe = (CIP'CIP - Cis induct) v i/A [*170-3132-38. (*276-01)]

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 *276'122. F:CG'Pe-.,E~v1.).C'p,, v cG'6= Cl'C'p [*276-121. *274-17] *276-123. F: C'Pr'.. e Cis induct. P6 [*276-1121] *276'13. -:CG'P,E 0v1.). NcPC'P- +, Nc'(C'P6= 2Nc'ICIP [*276-122. *116-72] *276-14. F: Pefl.). POESer [*251,36.(*276-01)] *276-2. F:P ew.).(C'P -)"(Cl induct'C'P- t'A) e ~rn ed'P,6 Dem. F.*24-492.) F.(C,'P-)"(CIlinduct'C'P-tifA) smn(Cl induict'IC'P -t'A) (1) F. (1).*274-27.:)F: Hp.). ('P -)"'(CIlinduct'C'P - tA) E, (2) F. *200-361. *263-47.) F: Hp. aP6/3..ze a - /3. a A P'~z = n P'z. ey = / v P'zJ. ly AP'z= /8nP'z = a^P'z.z ea- y.yr',Cls induict. [*276-11.*170-16]). aPqy. ryP0/ (3) F. *263-47. DF:Hp (3) D.).'P - y eCls induct (4) F. (3). (4). *276-11.-) F:Hp.aP6/3.:).(Hry).C'P-, yeClsinduct.aPory.ryP0$ (5) F. *120-71. Transp.) F:Hp.a ECl inductP'IP - t'A. (C'P - a)e Cis induct (6) F. (6). *276-121.. F: Hp. D. (C'IP -)"I(Cl induct'C'P - vIA) C CG'P6 (7) F. (2). (5). *271-1. (7). D F. Prop The following propositions constitute the proof of P e fIinfin.). Po eDed (*276-4). *276-3. F: E! PmIX.: aE TP'X..aeX. P'x e a: Pm'X= minp'(s'X - p'X) [(*276-03-04)] *276'301. F:Pe&1.XCCl'G'P-t'A.Xc —e~v1.:).E!Pm'6X.E!Tp' Dem. F.(1).Transp. *40-23.) F: Hp.). H! s'c- p'#. [*250-121]).E! Minp'(s'K - p'K): D F. Prop *276-302. F:E! Pm'X.).Pm'XCEp'Tp'X -p'X, [*276-3] *276-303. F. Tp C A. (Tp)P0 C A Dem. F.*276-30.)F:/ttTpx.:).,4x(2 F. (1). (2). *201-18.) F. Pi-op

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 *276-304. F:.I(pKI.),C."C6a,+."~~ [*276-302-3O3] *276-305. F-. A (Tp, K) EQ 1 [*258&201. *276-303] *276-31. I-:PoQ.a[!-X.XCC1'G'P-t'A.x,,eD'TP.:). x E 1. sX =

p'X= t'X [*276-301.Transp] *276-32. F:.IPEf.X~c~El.XCD'Po0.): Pm'x E6p' Tp'X - p6X: C X..a A JJ'Pm'X = p'X A P'Pm'X Dem. F.*276-301.)F:Hp.).E!Tp'X,.E!Pm6X. (1) [*276-302] D. Plrn'X le p'1fpX- 6- p (2) F.(1).*276-3.)F: Hp.).P'PmXASX~P', Pm',XAP'X (3) F. (2). (3). D F. Prop *276-321. F: Hp*276-32.aETp'X./3,eX-Tp,'X.). aP6/3 Dern. F.*276-332.) F:Hp.). Prtn'X C -,3.CLA P'fPm'X /3 A P'Pm'X. [*276-11] D. aP0/3: D F. PI-op *276-322. F: Hp*276-32.tke(Tp*A)'X.aclE,8t./eX-pt.). aP6/3 Dern. F(1). *276'321. *258&241.:)F. Prop *276-33. F:Hlp*276'32. g! p'(Tp*A)X.).tfp,'(Tp*A)'x = mmn(P,,yx Dem. F. *276-31. *258&231. D F: Hp. D. p'I(T*A)'XE1I (1) F. (1). *276-322.) F:Hp.aexX-p'(Tp*A)'X.). t'p,'(Tp*A)'X} Pea (2) F.(l). (2). DF. Prop *276-331. F: Hp*276-32. 2!p'(Tp*A)'x.. E! min(P0)'k [*276-33] *276-34. F:Hp *276-32. pTPX. /~ CD'Tp.).(Pm',X) P(Pm'pO) Dem. F. *276-3.)F: lip. D. Pm6AX = minp'(s'X - p'X) (1) F. *276-3'304. D F: Hp.)D. Pm'uE (s'X,-p'X) (2) [*13-12] D).Pm'xt+Pm'fa (3) F.(1.(2 -(3.).Prop R. & W. III. 15

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226 SERIES [PART V *276-341. F:. Hp*276-32.p'(Tp*A)'X= A.): PM"fl(Tp*A)'-X C P""Pm""(Tp*A)'X. Pm""(Tp*A)'"X, c~ CIS induct: Dern. F. *258-231. *276-301. F. (l).*261P26. Transp.:) F. Prop *276-342. F:Hp *276-341. X {A (Tp,/K)}L. E! Prn'p Cy (PM,('X) P (P1),ncl) Dem. F. *276-3.) F:: Hp::pC(Tp* A)'K.-,! p.~!p'lp:):. Pm'p'p es'p'p -p'p'p:. [*40-1*11-2]):. (a.D,:(a). aep'p. Pm'pp pca a: (ga.a)p. ame P,,pp E e a F. (1). *276-302.) F:. Hp (1").): Tp'X e p. X e p.)k. Pm'X ep'T'. P~pp e'pX: [*13-12]): TP'X E p. X Ep.:)A. PMT'X + Prn'P'P (2) F. (3). *276-34. *258-241. D F. Prop *276-35. F:.Pe.fLKCD'Po.g~bc.p'(Tp*A)'K=A.:): PCT -+ 'SPP~TT XC(Tp*A)'K - t,'A.. Pm'X e p'T riPPnT' Dem. F.*276-341.)F: HP. Xe(Tp*A)'K - t,'A.).E! Tp'x. [*276-302-34] Pm'X ep'Tp'x A P',Pm',Tp'x:) F. Prop *276-351. F: Hp*276-'35.).Pm,",(Tp*A)'KcC Ptl'K Dem. F.*276-35 D.(*276-05).) F: Hp.XE (Tp*A)'K-t'A.).Pm'XEPtj'i (2) F. (1).(2).) F. Prop *276-352. F:Hp*276.35.). Ptl'Kce~eClsinduct [*276-351P341] *276-353. F:Hp*276-35.Xe(Tp*A)'K.X[A(Tp, K)}u.ae/u.L). p' APm'X - pmpx A P'P rm'x = a A P'Pm'x Dem. F. *276-304.)F:Hp) aeX (1) F.*276'35-31. Transp) F: Hp.)E!Pm'X.XrsE 0 u1 (2) F.(1).(2).*276-32.:)F: Hp:).p'XAP'Pm'lXaAP'Pm'X (3)

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SECTION F] ON SERIES OF INFINITE SUB-CLASSES OF A SERIES27 227 F. (3). (5).) F. Prop *276-354. F:Hp*276-35.xE(Tp*A)'K aeX.). Ptl'K A Pp'PM X=])'X A-P'Pm'X = a P'PmX Dem. F. *276-353. DF: Hp. g! u. X{A (Tp, K)IFL.)D. 1)'I A] ~Prn'lx = jptx A PIPm'~X. [*22-47] D. (p',L A% P'Pm' lk) Ai P'Prx C p'X A P'Pm'X (1) F. *276-353. D F: Hp.u J A (Tp, Kc)} X. D. A~t fp'Ptaci = -p'X APpm [*276-342] CP'tXArPIPn' (2) F. (1). (2). *276-305.)D F: Hp., pE (Tp* A)'IK - t'A. D. (p'jLLA P'Pm'FL) A P'Pm'X C p'X A P'Pm'X (3) F. (3). *276-32. (*276-05). D F. Prop *276-355. F:Hp*276-35.aeK.:).(2x).Xe(Tp*A)'K. a eX. Plal'x ca Demn. F. *40'1.)F:. Hp.)D (ak). XEc (Tp*A)'K.a eX: [*276-305]):a)X TpA'ceX,{ATp)X.)aEL (1) F*40-1.. DF:.{A (T, K)} X.),. a: X = PA (T, K)'X:)D.a EX (2) F. (1). (2).

Transp.)D F: lip.:). (aV6)~L X(Tp*A)'xK. X = Tp'~u. aE,i p. ae X. [*276,3]:). (gj/).,.Le(Tp*A)'K. aEFL.Pm'FL(%)a:)F. Prop *276-36. F:Hp *276'35.a6EK.). (Pti'K) Pea Dem. F.*276-351-355-354.)D F: Hp.)D. (ax). X6 E(Tp *A)'Kc. Pm'XE16PtI'K - a. Pti'CKAnP'Ptn,'X a A-iP'Pm'X. [*276-352] D. (PtI'CK) Pe a:) F. Piop *276-361. F: Hp *276-35. D. K C PO'rPt~CK [*276-36] *276-37. F:Hp *276-35.(Ptl'c) P013.). (2a).a EK.aP0,8 Dem. F*276-11D)F: Hp.D. (az). z ePt~I'K-,8. P CJKA PIZ =13APIZ. -4) [(*276-05)] D. (Hz, X). XE (Tp*A)'K. ZE EpX A, P'IPm'x /8. PtI'K A P'Z =8 / P'z. [*276,354])(Hz, X, a).VE(Tp *A)'K.ae6X. za -/3. P'~z C P'Prn"X. a~ A P,'Pm'X =13 A P'Pm'X. [Fact.*276-304]).(aIz,a). ae/.zea-13.13AiP'zCa. [*170-11])(3a).aE K. aPO13::)F. Prop 1 5-2

228 SERIES [PART V *276-38. F:PE&E.KCD'P,.a! K. p'(Tp*A)'K= A.) PI1jC = tL (P,6)'K [*276-361-37] *276-381. F: Pe f2.KCD'IP6.! K.p'(Tp*A)6K= A.:).E! t1 (PO6yK [*276-38] *276-39. F: Pe E2 K C D'1PO. a! K.). E! limin (P,)'K [*276-331P381] In the following proposition, the only reason why P has to be infinite is in order that Po may exist; for "Ded " was so defined as to exclude A *276-4. F:P ef2 i~n n)Po EDed Demi. F.*276-12-1.*207-3.*205'18.:)F: Hp.:). lirninpA= G'P. limrnip't'A = A (1) F.-*206-7.)F: Hp.KC CPo. A eK. K +tA.) prec (P,)'K = prec (PO)'(K -t 'A) (2) F. *205 1 92.) F:H p (2).)D. mmi (P,)'K = m i (PO)'6(K - t'A) (3) F.(2).(3).:)F:HP (2).).lirniin(P6)'K = limin (Pe)'(K -t'A). [*214-12-14]) P6e Ded:.). Prop *276-41. F: PeO). P6EO [*276-2-414.*2751]' *276-42. F:Pe co C.'P6e 2KO Dem. F. *276-13. *274-27.)D F: Hp.). Nc'C'Po 0~0No2Wo(1 F.*276-2. D F:Hp. D.(Hu).Nc'G'PO=~00 [*123-421]).- Nc'G'P, +,, N = Nc'C'P6 (2) F. (1).(2).:) F. Prop *276-43. F. CG"O - M Dem. F.- *276-42-41.) F: a! c.) a! G"O0 2No. [*100-42.*275-33.*152-71] D. C6"0 = 2No (1) F.*275-11.*263-101. F: co=A.). O=A (2) F.*263'101.*116-204.)F:wo=A.).2No=A (3) F. (2).(3).)F:co=A.:). G"O=2XO (4) F. (1).(4).:) F. Prop The propositions proved in the present number are capable of being to some extent generalized. Also we can prove F. 9 = (wexp,. coA)+1

SECTION F] ON SERIES OF INFINITE SUB-CLASSES OF A SERIES 229 For this purpose, we prove first that if P, Q are well-ordered series, PQ is Dedekindian (except that if ~ E! B'P, PQ has no last term); i.e. we prove P, Q e Q. X C' CCP. 2 [! X.:D. E! limin (PQ)'X. For this purpose, assuming X CC'P. a! X, put Q"X = minQ' ^ (0"y (" e 0 V 1), Tp'x = X n, M {M'QmX = minpv"k'Qm'XJ, A =X C X. + X), (PQ)'x = 'N {(aH). e (Tp*A)'x. N = (p'p) r Q'Qm'i}. We can then show, by steps closely analogous to those in the proof of PoeDed, that we have a! p'(TpA)'X. D. t'p'(Tp A)'X = mIin (PQ)'X,,!!p'(Tp*A)'X.. (PQ)'x = prec (PQ)'X, whence, in either case, E! limin (PQ)'x. Hence we have: P, Q e Q. E! B'P.. PQ. e Ded, F: P, Q e fn. E! B'P. Z e C'PQ. D. PQ - Ze Ded. We have therefore F. (co expr co)+ i C Ded. We now have to prove Q (co expr co) + i. c. D! Ko n med'Q. For this purpose, it will be sufficient to prove P e co.. X! No n med'(PP). The Ko in question will be the class of those members of C'(PP) in which, from a certain point onward, the correlate of every member of C'P is B'P. We have M (pP) N. -: M, N e (C'P T C'P) a'C'P: (q{x). x (C'P. MrP'x = NP'x. (M'x) P (N'x). Now consider the relation L = M rP*'x: y $ P,'x: (,tB'P)T P'P'x, where (M'P1.') Py. Then M (PP) L. L (PP) N. Also L has B'P for the correlate of every term after P/'x. Hence it is determined by the correlates of the terms up to and v v including P1'x. Thus, putting z = P1'x, we have to consider the class of relations = {(az). z e (['P. X e 1 -- Cls. ('X = P*'z. D'X C CP}.

230 SERIES [PART V If X e p, X w (L'B'P) T P'maxp'a'X is a member of C'PP. We have therefore only to show that JL e o. To show that pu e ow, we observe that if X e,p, D'X and (I'X are both inductive classes; hence each has a maximum. Let X and X' be two members of /u, and let us put x = maxp'D'X. x'= maxp'D'X'. y = maxp'(['X. y'= maxp'I'X'. If x= -p and y=vp, put x+py=((+c v)p. Then put X before X' if (x +py)P (x' +py'), or if x +py = x' +py'. yPy'. But if x +py = x' +py' and y = y', i.e. if x - '. y = y', take the immediate predecessors of x, y, x', y' in D'X, aPX, D'X', CI'X' respectively, and apply the same tests to them, and so on, until we come to a difference. In this way, we obtain an arrangement by last differences (in a slightly extended sense), and this arrangement is easily shown to be an w. Hence,Le N. Hence the class v = 7 {X) X e. = X (LtBP) P'maxp'G'X is an o,;and we have already shown that it is a median class of C'PP. Hence F: P e co. D.! m ed'(PP). The same class will be a median class of PP -) Z, if Z e C'PP. Hence F: P e w. Z e C'PP.). [! No, med'(PP * Z). Hence, by what was proved earlier,: P e co. Z e C'PP. D. (PP -* Z) e 0, i.e.. (wo exp, o) 4 i = 0.

PART VI. QUANTITY.

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SUMMARY OF PART VI. THE purpose of this Part is to explain the kinds of applications of numbers which may be called measurement. For this purpose, we have first to consider generalizations of number. The numbers dealt with hitherto have been only integers (cardinal or ordinal); accordingly, in Section A, we consider positive and negative integers, ratios, and real numbers. (Complex numbers are dealt with later, under geometry, because they do not form a one-dimensional series.) In Section B, we deal with what may be called "kinds" of quantity: thus e.g. masses, spatial distances, velocities, each form one kind of quantity. We consider each kind of quantity as what may be called a " vector-family," i.e. a class of one-one relations all having the same converse domain, and all having their domain contained in their converse domain. In such a case as spatial distances, the applicability of this view is obvious; in such a case as masses, the view becomes applicable by considering e.g. one gramme as + one gramme, i.e. as the relation of a mass m to a mass m' when mn exceeds m' by one gramme. What is commonly called simply one gramme will then be the mass which has the relation + one gramme to the zero of mass. The reasons for treating quantities as vectors will be explained in Section B. Various different kinds of vector-families will be considered, the object being to obtain families whose members are capable of measurement either by means of ratios or by means of real numbers. Section C is concerned with measurement, i.e. with the discovery of ratios, or of the relations expressed by real numbers, between the members of a vector-family. A family of vectors is measurable if it contains a member T (the unit) such that any other member S has to T a relation which is either a ratio or a real number. It will be shown that certain sorts of vector-families are in this sense measurable, and that measurement so defined has the mathematical properties which we expect it to possess. Section D deals with cyclic families of vectors, such as angles or elliptic straight lines. The theory of measurement as applied to such families presents peculiar features, owing to the fact that any number of complete revolutions may be added to a vector without altering it. Thus there is not a single ratio of two vectors, but many ratios, of which we select one as the principal ratio.

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SECTION A. GENERALIZATION OF NUMBER. Summary of Section A. In this section, we first define the series of positive and negative integers. If a is a cardinal, the corresponding positive and negative integers are the relations +c/t and -c,, or rather (+c,) C (NC induct- t'A) and (-c /) (NC induct- t'A). (It will be observed that a positive integer must not be confounded with the corresponding signless integer, for while the former is a relation, the latter is a class of classes.) We next proceed to numerically-defined powers of relations, i.e. to R", where v is an inductive cardinal. We have already defined R2 and Rs, but for the definition of ratio it is important to define Rv generally. If R e 1 -- 1. Rpo C J, we shall have RV = RB, and if R e Ser, we shall have (R)v = Rl. But these equations do not hold in general, and in particular if R C I and v $ 0, R = R but R, = A. After a number devoted to relative primes, we proceed to the definition of signless ratios, thence to the multiplication and addition of signless ratios, thence to negative ratios, and thence to the generalized addition and multiplication which includes negative ratios. (In the case of ratios, signless ratios are identical with positive ratios. This is possible because signless ratios, unlike signless integers, are already relations.) We then proceed to the definition of real numbers, positive and negative, and to the addition and multiplication of real numbers. At each stage, we prove the commutative, associative, and distributive laws, and whatever else may seem necessary, for the particular kind of addition and multiplication in question. Great difficulties are caused, in this section, by the existence-theorems and the question of types. These difficulties disappear if the axiom of infinity is assumed, but it seems improper to make the theory of (say) 2/3 depend upon the assumption that the number of objects in the universe is not finite. We have, accordingly, taken pains not to make this assumption, except where, as in the theory of real numbers, it is really essential, and not merely convenient. When the axiom of infinity is required, it is always explicitly stated in the hypothesis, so that our propositions, as enunciated, are true even if the axiom of infinity is false.

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*300. POSITIVE AND NEGATIVE INTEGERS, AND NUMERICAL RELATIONS. Summary of *300. In this number, we introduce three definitions. We first define " U" as meaning the relation which holds between a +c v and /A whenever s and v are existent inductive cardinals of the same type, and v $ 0, and, +,c v exists in this type. Thus U is the relation "greater than" confined to existent inductive cardinals of the same type. The definition is: *300 01. U= (+c l)po (NC induct - tA) Df Then if 1a is an inductive cardinal which exists in the type in question, U, and U,4 are the corresponding positive and negative integers, where " U," has the meaning defined in *121. It will be observed that OU,,z, so that U, exists, when,a exists in the type in question. We prove (*300-15) that U is a series, and (*30014) that its field consists of all existent inductive cardinals of the type in question, its domain consists of all its field except 0, and its converse domain of all its field except the greatest (if any). If the axiom of infinity holds, C' U consists of all inductive cardinals. It will be observed that U arranges the inductive cardinals in descending order of magnitude. The reason for choosing this order instead of the converse is that U is less required in its serial use than as leading to the functional relations U.,4. As explained at the end of Part I, Section D, there is a broad difference between functional and serial relations, and this produces, where one relation (or its derivatives) is to have both uses, a certain conflict of convenience as to the sense in which the relation is to be taken. Considered as arranging the integers in a series, U would naturally be defined so as to arrange them in ascending order of magnitude, as was done with "N" in *123. But considered as functional relations, it is more convenient and more natural to take (say) +r 1 as the relation to start with, and - 1 as its converse. Thus we want,u U v when / = i +c 1, i.e. we want U1'v = v +~ 1; and this requires the definition of U given above. We prove in this number (*300-23) that U is well-ordered, and (*300-2122) is either finite or a progression. We also prove (*30017-18) that, if, is any

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University of Michigan Historical Math Collection: Principia mathematica, by Alfred North Whitehead ... and Bertrand Russell.

236 QUANTITY [PART VI typically indefinite inductive cardinal, tA and L' + 1 will belong to C' U if U is taken in a sufficiently high type. Our other two definitions in this number define two classes of relations which are of vital importance in the theory of ratio. We define numerical relations, which are called " Rel num," as one-one relations whose powers are all contained in diversity, i.e. we put *300 02. Rel num = (1 -e 1) n R (Pot'R C R1'J) Df We thus have (*300'3) F: Re Rel num. =. Re -*1. Ro C J. It will be remembered that the hypothesis Re(Cls —l)u(I —+Cls).RpoCJ played a great part in *121, and in all later work which depended upon *121. When both R and R fulfil this hypothesis, we have ReRel num, and vice versa. We prove (*300'44) that if ac is an inductive cardinal not zero, and P is a series, then Pa is a numerical relation, and so is P,. If P is an endless well-ordered series, finid'P (i.e. the class of relations P,) is what (in Section B) we shall call a vector-family: P, is the vector which carries a term a steps along the series. In order to be able to deal with zero, we have to consider the application of ratios, not only to such relations as are numerical in the above sense, but also to relations contained in identity, because a relation contained in identity may be regarded as a zero vector, so that (e.g.) if P is a series, I r C'P will have a zero ratio to P, if a is an inductive cardinal other than 0. We therefore introduce a class "Rel num id" consisting of numerical relations together with such as are contained in identity; these may be called numerical or identical relations. They may be defined as one-one relations whose powers, other than R0, are contained in diversity, because, if R C I, there are no powers other than Ro. Thus we put *300 03. Rel num id =(1 - 1) n R (Potid'R - t'Ro C RR1J) Df and we then prove *300'33. F. Rel num id = Rl'I v Rel num For the application of ratio, it is important to know under what circumstances there exists a numerical relation R such that Ra is not null. We prove (*300'45) that if a is an inductive cardinal, and P is a series of - +~1 terms, then (B'P) P,(B'P). Now we also prove (*300-44) that if P is a series, and R =P, Pa, =R and R is a numerical relation. Hence it follows, by *262-211, that if a- + 0 and a is a class of a +c 1 terms, there is

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SECTION A] POSITIVE AND NEGATIVE INTEGERS 237 a numerical relation R whose field is of the same type as a and for which R, exists. Remembering *300'14 (quoted above), this proposition is: *300'46. F: -e C IU- 'O.. (gP, R). P e (+, 1)r. R = P. R e Rel num. t'C'R = to'o. (B'R) R (B'R) We have conversely (*300'47): R e Rel nuLm.! R,... a c NC ind.! (a +, 1) nr tC'R. - n t'C'R e (' U, where " NC ind" has the meaning defined in *126, i.e. " a e NC ind " means that a- is a typically indefinite cardinal. The number ends by propositions proving (*300-52) that U. is a numerical relation, that (*300'57):(U), A (u) g. n (x,,1, c [7. x, = X7 Xo t,, and analogous theorems. *300 01. U = (+o 1)po t (NC induct - t'A) Df *300-02. Rel num = (1 -- 1) n R (Pot'R C R1'J) Df *300-03. Rel num id = (1 — + 1) n R (Potid'R- tRo C R1'J) Df *300'1. F:,Uv. =. (+c l)po v. A, t e NC induct - L'A [(*300-01)] *300'11. F:. UXv.=: /a, N e NC induct - 'A: (X). X e NC induct - ' O. = v +X: =: J,v e NC induct - t'A: (aX). X 0.= v +o X: A: v, e NC induct - 'A: (aX). X e NC - 'O.,u = V +o X [*300-1. *120-42'428-462-452. *110-4] *300-12. F:, Uv. -.,A, v e NC induct - tA. v <,.. A, e NC induct. < /,. -., NC induct.

http://quod.lib.umich.edu/cgi/t/text/text-idx?c...stmath;rgn=main;view=text;idno=AAT3201.0003.001 (120 of 246) [5/26/2008 7:31:50 PM]

v < A [*30011. *117-3. *120-42. *117 26. *110-6. *11715. *120'48] *300-13. F. U J [*300-12. *117-42] *300'14. F. C'U= NC induct - 'A. D'U= NC induct- 'A - '0. ' U= NC induct n ^ (a!v +c 1) = v (v +0 1 e NC induct - L'A). B'U= 0 [*300-12. *117-511. *120-122. *101-241. *120-429-422] *300'15. F. Ue Ser [*300-13. *120-441] *300'16. F: a e Cls induct. D. Noc'a e C' U n t2a. NoC'a e C'(U t2'a) Dem. F. *120'21.: F: Hp. D. Noc'a e NC induct (1) F. *103-13. D F. Noc'a + A (2) F. *10311. D F. Noc'a e t2'a (3) F. (1). (2). (3). *30014. D F. Prop

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238 QUANTITY [PART' VI *300-17. F:,ttENCind.).(Hja).qzt'taECtU.,aEGI(U~t2tla) Dem. [*103-34]:). (ga).a~ECIs induct.,uttr = N,,c'a (1) F.(1). *300-16.) F:Hp. D. (Ha). p nt'a eC'U. (2) [*65-13]:). (H) i 6C' U.,tC t'a. F-. (2). (3). D F-. Prop *300-18. F: u eNC ind.). (Ho-). 2 e C'(U ~ t2Co-). (~1) tT 6C EGU /EU (1 U t2%T-) [*126-13-15. *300-17-14] *300-181. F: te NC ind. p nt'aE O'U.) Ak t2a E C'U. (i1 +c 1) A t2lae (J'U. pt A t2lle(aE1iU [*126-23. *300-14] *300-2. F:Infln ax.) U= NpO Here N has the meaning defined in *263-02. Dem. F. *300'1.*1251.) F:. Hp.:),ttUv.. v E NC induct. Itt (~, 1),O v. [*1 20-1.*91-574] *v (~- 1)* 0.)P(O 10v. [(*263-02.*120-01)].VNO pt:.) F. Prop *300-21. F:IInfin ax.). UEw [*300-2. *263-12] *300-22. F: Infin. ax.) U e f2 induct Demn. F. *1 25-1624. Transp.:) F: Hp.). G'U e Cis induct(1 F. (1). *300-15. *261-32.) F. Prop *300-23. F.- U ef~ [*300-21V22] *300-231. F:ptU~v.=-.pa,veNCinduict-t'A.,a=v+01I. /,tte NC induct - tA. /,t = vi +0,,1. Pe NC induct - tA- tI0. v =,tt I~1 =VENCiflduct-tffA.v=,tt- I Dem. F. *300-15-12. *201-63.) [*120-429]: t, v E NC induct - t'A. v <p1,: v +0 ~, at.>, v~01 [*117-25] =:/p,veNCinduct-t'A.pa=v+01 (1 F. (1). *1 20-422-424-423.:) F. Prop

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SECTION A] POSITIVE AND NEGATIVE INTEGERS 239 *300-232. I- At E NC induct.). U,= (+p,) I (NC induct - t'A). U,= (-, u C (NC induct - t'A) For the definition of U, see *121P02. Dem. F.*121-302.*300K15.) F:p U a-.. a-eC'U. p=o. [*300,14.*110-6]. p, a- E NC induct - t"A. p = a- +, (1) F. *260-22-28. *121-332.) F: (-i- (+,,) ~ (NC induct - 'A).). U,+,, = (+,tt) C (NC induct - tfA) I U, [*300-231] = (+,, tk (NC induct - V'A) I (+, 1) C (NC induct - V'A) [*120-45452] = {+O (/, +e 1)j ~ (NC induct - t'A) (2) F.(1). (2). Induct.) F.Prop *300-24. F: pe NC induct.).D'U,, = U*'At = NC induct n V (v) [*300-232. *117-31. *120-45] *300-25. F jt E NC induct.). B U,= U't= NC induct n U)(v<A)= U(0 U- p-) [*300-232-24-12] *300-26. F: e C' U. --. A U, U_ (C' U) [*300-232-14. *110-6] Here the At in " U," is of higher type than the FL in "IL E 0' U," because the interval U(0 p, u) is composed of members each of which is of the same type as At *300-3. F: Re Rel numn. R e1-,1 RPO C J.. R e1-+1.Pot'R C Rl'J [(*300-02)] *300-31. F: R e Rel num id.. R e I -+I. Potid'R - t'Ro C Rl'J [(*300-03)] *300-311. F: R C I. E. RR = R.. R = I C'R Dem. 4- - F.*201-13-18.)F:.RC I.):xeC'R.).R*'XAR*'x=t 'x (1) F. (1). *121-11.)F: R CI.)I rCR CRo. [*121-3] R. R0=Ir C'R. [*72-92]) = R =1 r

61R (2) F. *121-3.)F:R0=R.. R C 1 (3) F - (2). 3:) F. Prop *300-312. F:R C I. -). Potid'R? = t'1R = tf]R0 [*300-311. *5072. Induct] *300-313. F: RERelnum id.). R*~-:R, CJ [*300-31.*9155]

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240 QUANTITY [PART VI *300-32. F:REReltiumid.:).R0)=IrC'B Dern. F.*91P35) F -ItrC'REPotid'R -RIlex'IJ (1) F.(1).-*300-31.) F. Prop *300-321. F:RERelnuimid.R +j=R,.:).RCJ.ftjjR [*300-31] *300-322. F:R CJ.).RP A R0= Deni. F.*1 21-3.)F: xRp0y -x 4zy.).xRoy) (1) F.*50-24. F:. Hp.): - (xRx) - (2) [*121-103.(2)])R (xi- 4x) +'tx. [*121-11])%(XR,)x) (3) F. (1). (3). DF. Prop *300-323. F:-RcRelnumid. R#Ro.:). RP0CfJ Dern. F.*300-321-322.) F: Hp.:). RP0AR0=A. [*300-32]:). RP0 A I G'OR = A:)D F. Prop *300-324. F:.1?eRelnuimid.):-RC-I.v.1?ERelinum Dem. F. *300-311-323.)DF:. Hp. D:JC I. v. Rp. CfJ (1) F. *300-32. D F: Re Rel num id. RPO CfJ.:). Potid'R-t'B0=Pot'1? (2) F.(2).*300-31.:)F:ReRelnuimid. Rp0CfJ.). Pot'RCR1'J (3) F.(1).(3).*3003. F.Prop *300-325. F:R CI.)R eRel num id Demn. F. *300-312. F: Hp.. Potid'R - t'RO =A (1) F. (1).-*300-31.) F. Prop *300-326. F:R eRel num.)D. ReRelnum id Dem. F.*121-3.*300-3.:)F:Hp.:).R(,ePot'R (1) F. *121-302.*300-3.:) F: Hp.:).1R?,=IrG'6R (2) F. (3). *300-3,31.:)F. Prop *300-33. F.Rel num id= R1'I vRel num [*300-324-32.5-326] *300-34. F. A e Rel num [*300-3. *72-1] *300-4. F. Rel num = Cnv""Rel num [*300-3. *91l522] *300-41. F. Rel num id = CnV"'Rel num id [*300-31. *91'521]

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SECTION AI SECTIIN A]POSITIVE AND NEGATIVE INTEGERS24 241 *300-42. F:1? ERel num.) Pot'lR CRel num Dem. I- *91-6. *92-102.)D I- R e Rel num. P e Pot'] R). P e 1 -+ 1. Pot'P C R1'J. [*300-3] D. PeRel num: DF. Prop *300-43. I-: R eRel num id.). Potid']?C Rel num id Dem. F. *300-325-312. D R CI1. D. Potid'R C Rel num id (1) F. *300-32 ". DF.I r OR Rel num id (2) F. (2). *300-42-326. D F: 1Re Rel num..). Potid'R C Rel num id (3) F. (1).(3). *300-33.) F. Prop *300-44. F:.Pe Ser.o-,ENC ind.) Pa, Pa E Rel nurn id: a- # 0.) P, = (Pi)a. P~, P, E Rel num Dem. F.*121-302.*300-32'.:)F:Hp.o-=0.:).Pa,P,,ERelnumid (1) F. *260-28. DF:Hp. o-+0.D.P.=(P1),. (2) F. *300-3. *260'22.)F:.llp.):Ple Rel num: [*121P5.*300-42] D): a- # 0.. (PI), E Rel num. [(2).*300-4] Par)P~,PERel num (3) F. (1). (2). (3).:) F. Prop *300-45. F:o-eNCind.Pe(a-+c1)r.).(B,'P)Pc,(B',P) For the definition of (a- +,, 1)r, see *262-03. Dem. F. *262-12. D F: Hp.:). P cfl. C'Peoa-+1 [*202-181. *261V24] D. (B'P) P. (B'P): D F. Prop *300-46. F:o-,ECPU-t'0.D. (HP, R). P E (a- +,c 1)r. 1? = P,. R c Rel num. tPC'R = t0'oa-. (B']?) R~, (B,']) Dem. F.*300-14.) F:Hp.). (ia). a eCis induct.t'la =t0'o-. a E - +01. [*300-45] - (HP). P E6 (a- +0l)r. tP'P = to'a-. (B'fP) P., (B'P). [*300-44.*2 61-22]).(aP, R). Pe (0-+cII)r.]?R= P, R1?Rel num. tII,= tola-. (B']) R0, (B'.R): D F. Prop &W. III. 16

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242 QUANTITY [PART VI *300-47. F: B eRelnum. ft!R,. D. 0- E NC ind. a! (a- +0,,1) ni t'C'R. a- n t'G'B E (I' U Dem. F.- *1 21 11.) I-: Hp.)D. (gjx,y)..1 (x i-iy) E a- +01 I. [*121-46] D. a -~ e NC ind. a! (a +0 1)Ant'CR. [*120-422.*300-14]:). a-,NCind.a~!(a-+~01)nAt'C'R. a-A t'C'R e (P U: D F. Prop *300-48. F: RCI. vto.:). R= Dem. F. *300-312-311. *91P55. DF: R GI. D.R* = I COR F. (1).*121-103.:)F: RBCI.:).R?(x~-4y)=CO'R At'x At'ZI [*117-222] D. v ~1 <, Nc'tx. [*117-54.*120-124] D. v +o1 = 1. [*110-641.*120-311]) =0 F.(3). Transp.)F.Prop *300-481. F: Re Rel num id vt+0)(R0),=A (R1,)0,C R Demn. F. *300-32-48 D) F: Hp. (R).(B), = A F. *300-43-32. D F: lip. D. (R,~)0 = I r C'BR,. [*121-322.*300-32]).(Re,), C Ro F. (1).(2).) F.-Prop (1) (2) +0,1. (3) (1) (2) *300-49. F:ReRelnum. Ar~ePot'R.:).C'BR-~EClsinduct Dem. F.*1215.:)F:.HUp.:): v eNC induct.). f! R,. [*121-11]:).! (v +0,1)AnC1'G'R:.)DF. Prop *300-491. F: (as).RBe Rel num.A, EPot']?.). Jfin ax [*.300-49] *300-5. F. Ue Rel num [*300-15-44] *300-51. F. U, E Rel num id [*300-15-44] *300-511. F.- U = (U1), [*300-21-22. *263-491] *300-52. F: p E NC ind - tf0.)D. U,,E Rel num [*300-15-44] *300-53. F. (x0,1) ~ C'UEcRel num id [*300-325. *113-621] *300-54. F: Infin ax. k.~DIU- t'1.) - (x01ap) ~ D'UE6Rel num Dem. [*117-47-42]:): {(x0, I) ~ D'fU} C JI F.(1).(2). *300-3)DF.Prop (1) (2)

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SECTION A] SECTION A]POSITIVE AND NEGATIVE INTEGERS24 243 *300-551. I-:!RPABR.E.~R,,.p [*300-55] *300-552. F:R E Rel num.).(RI), C Re,, Dem. F. *121-36. DF:Hp.4:,veNCind-tf0.D.(Rt),=Rj,, (1 F.*300-481.)F:Hp.4:=0.v+O.).(Re), =A, (2) F.*300-32-311.*11.3-602.)F:Hp.4:=O.v=0.).(R4),=R4l,,, (3) F. *300-47.) F: Hp. v~4,vENC ind).)(RI),= (5) F-. (1). (2). (3). (4). (5).) F. Prop *300-56. F: ReRelnum.ft(R1?,A(R4,)A. D xC: ~=',q x01U. (4: X0 v) A t'U'R 6 (ItU Dem. F. *300-552.) F: Hp. D. f! RjevAR7,~ (1) F. (1). *300-55.)DF. Prop *300-57. F:ft!(Ut) VA(U,,.:).4:xoveC'U.4:xov=flx0FL4 Dem. F.-(1).*300-26. DF. Prop By *300-56, we have, with the above hypothesis, (4: xO v) A t'C' U e 01' Uf. But here the U in G 1'U is of higher type than the U in (4: x 0 v) A t'C' U or in the hypothesis. In the type of the U in the hypothesis, we have 4: x,,' e 0' U, not necessarily 4: xO v E 01' U. *300-571. F:.,qDU::t(I~AU7I-~O'cl.XV' "~ Dem. F. *300-26. D F 4xOvEOU. 4:x,,.v =, Xc).(4:xO) t Utx>A U,7x.C4j0 (1) F.- *300-32.) F: Hp. Hp(1). v= 0.). (Ut)v= I ro GU,. [*300Q26]).0 {(UV} 0 (3) Similarly F: Hp. Hp (1). /a=0.:). 0 {(U44,j0 (4) F.*113-602. DF: Hp. Hp (1). v=0.:).,a= 0 (5) F.(I). (2). (3). (4). (5).) F: Hp. Hp (1).).f[! (U jj)vA (U,),, (6) F.-(6). *300-57.)FProp *300-572. F:4E'.: (~r~ WU [300-5714:~ 1 6-2

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*301. NUMERICALLY DEFINED POWERS OF RELATIONS. Summary of *301. In this number, we have to exhibit the powers of a relation R, i.e. the various members of Potid'R, as of the form RT, where ao is an inductive cardinal. We

University of Michigan Historical Math Collection: Principia mathematica, by Alfred North Whitehead ... and Bertrand Russell.

have already had R2= R I R and R3= R2 R. What we need is a definition which shall give R+,I = R, I R. Now RB is a function of R and or; thus we have to exhibit RB in the form S'a, where S will be a function of R. That is, we have to define the relation S as a relation of RB to a, and S must be such that, if it holds between Ro and a, it holds between R'+o1 and a +c 1. Thus we may take S as a sum of couples, such that if one couple is Ra 4 a, the next is (R I R) (ao +c 1), i.e. such that, if one couple is Q 4 a-, the next is (Q I R) 4 (a +c 1). Now (Q I R) I (a +c 1) = (I R) | (- 1)'(Q I a). Hence, since we want to have R~ = I 0 C'R, our class of couples is M [M {(I R) i(-o 1)1* (Ir CR)4]. Calling this class num (R), we may therefore put Ro= JsCnum (R)}'a Df. If we put (I R) 1 (-c 1) = Rp, the above definitions are num (R) = (Rp)*'{(I r C'R) 4 O} Dft, Rh = {'num (R)}'a Df. But the above definition of Rp requires some modification before it can be considered quite correct. With the above definition, we have Rp'(Q) = (Q I R) (a + 1) (1). Now since num (R) is defined by means of (Rp)*, and since the definition of R* contains the hypothesis R"1I C p, it follows that, if num (R) is to be significant, the relation -ol which appears in the definition of Rp must be homogeneous, so that, in (1), a and a- + 1 must be of the same type. Hence a, though typically ambiguous, cannot be typically indefinite;

 therefore, if the axiom of infinity is not true, we shall sooner or later arrive at o = A as we travel up the inductive cardinals. In that case, we shall have Ra-cl, (a -c 1) e num (R). (R(-ol R), A e num (R), (R-cl1 R I R) 4 A e num (R), etc. Now if (for example) R is a cyclic relation, such as that of an angle of a polygon to the next angle to the left, we shall not have RB-cl=Ra-cl R or Rr-oclR R-R-clIR. Hence s'num (R) will fail to be one-many, and RB will fail to exist. Hence it becomes desirable to restrict a- to cardinals which exist in some assigned type, i.e. to replace -o 1 by (-c 1) (NC induct - t'A), i.e. by U1. Thus we now put Rp= (R) U, Dft. But even this definition is not quite complete, because the type of U is not assigned. It makes some difference how the type of U is assigned, for if we take as the type of C'U a type lower than that of t'Noc't'R, we may find that our numbers become A before we have ceased to obtain fresh powers of R. For example, suppose the total number of individuals were four, and that these were a, x, y, z. Let us write x, (a, y,...) for x, a v x, y v.... Then consider the relation R = x 4 (a, y) w a 4 y w y (x, z). Then R2=x (x,y,z) u a, (x,z)wy 4 (a,y), R3x, (a, y, x, z)a (a, y) iy, (x,y,z), R =x 4 (y,x,z,a) ia, (y,x,z) y, (a,y,x, z), R5 x w (a, x,y,z) Ia, (a, x,y,z) y 4 (a,x,y,z). After this, R5 = R6 I R = R5 I R2= etc. But up to R5, each power of R is different from all its predecessors. If we take t'C'U= t'Noc't'C'R, C'U consists only of the numbers 0, 1, 2, 3, 4, and is thus inadequate to deal with R5. Hence the type in which we take U must be a sufficiently high type, which must increase with the type of R. Hence we take C' U in the type of t'Noc't'R, i.e. in the type of t3'R. This is secured by writing U t3'R in place of U in the definition of Rp. Hence the final definitions for RB are: *301-01. Rp= (R) 1i (U C t3'R) Dft [*301] *301-02. num (R) = (Rp,)'t(I [C'R) 4 (0 n t2'R)} Dft [*301] *301o03. R = { 'num (R)}' Df The two temporary definitions *301'01'02 are only to extend to the present number.

http://quod.lib.umich.edu/cgi/t/text/text-idx?c...stmath;rgn=main;view=text;idno=AAT3201.0003.001 (124 of 246) [5/26/2008 7:31:50 PM]

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246f QUANTITY [PART VI With the above definitions we have *301-16. F:,. kEC'Unt3'B.=E.E!R-" *301 2. F.RO =IrC'R R'=-B *301-21. F: vc G"PU n VtR. D.)Rv +cl = Rv I R *301-23. F: p +, v e Ct U n PIR. D. Rt+~o = Rg I RY = Rv I R,% *301 26. 1: P E Potid'R. =-. (go-). P = Ro I.e. the powers of R are the various relations RC. This proposition might have been not universally true if we had taken U in a lower type. *301 3. F-: R C I. a- EC'U n t3"'R.D. R7 = R = 1 = I r C'R Jt is largely for the sake of this proposition that we require powers of relations in dealing with ratio, rather than finid'R. For we have RCZ.0#O.D.14= A, so that R, does not give what is wanted if R C I. On the other hand (*301P41), if Re Rel num, we have R =R, if o- C'Uri t3'R. Thus as applied to numerical relations, R, may always replace Ro. We have, whatever R may be, *301-504. I:a,ve C'UA t2'C'R. v#0.). (R()v==RilXcv The importance of this number will appear in connection with ratios. *301-01. R =(R)J 1 (U, ~ t'R) Dft [*301] *301-02. num(R) =(R)*'t(I 0C'R)4 (0rnt2'R)1 Dft [*301] *301-03. R"= {B'num (R)j}'o- Df *301-1. F a- E (I'(U t3'R). D. R/'(Q 4 a) =,(Q 11) f(a- +Rl) A t2'R} [*55-61. (*301-01)] *c301-101. F: 0- e (I (U t t3'R). a-. (CC U n t3"R, -. 1 C U. 0- C t2"R [*63-5] *~301-102. F: e 1(U t3 3R) (aX). X e Cls induct, a! - X. R e t'"X. a- = Noc'X [*300-14. *103-11] *c301-103. F: o- e ('(U t'R). (HX) - X e Cls induct. - X. R E X. a- = N,c' [*301-102 - *73-71-72] *301-104. F: a E (I'(UG C t3'R). (a +, 1) n t2'R E NC induct - CIA [*301-101. *300-14] *301-105. F:I-e G'(UC tU'R). (Hr.) (X e Cls induct. Re X. - +, 1 = N,c't [*301-104]

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SECTION A] NUMERICALLY DEFINED POWERS OF RELATIONS24 247 *301-106.F: oGU t') (a)XesidutRet'. =Nc' [*301-104] *301-107.Fo- (UtR) oNid.RE(o+) [*301-106. *126-1] *301-11. F: a-6C E"(U ~t~'R). =E!R1pc(Q 4 e) [*30111] *301-12. F: Mcnum(R).).(gfP,o)PcPotid'R.a-eC'UA t3'R.M=P4,cr [*95-22] *301-13. F:P 0 enum (R).). P= I rG'R Dem. I -. *90-31. (*301-02).)D F:P 4,p cnum (R) -t f(IGR) 4, 0.) (P 4, p) [(Rp)* IRpJ 1(1 r (Y'R) 4, 01. [*30-33.-*301 1]) (P 4, a) (Rp)* (R 4, 1). [*300-24]:).,a== (1) F. (1). Transp.) F. Prop *301-14. F:P4,,u,,Q4, penuin (R).)P =Q Dem. F.*120-124. *90-31.)D F:IS 4, (tt + 1)1 (Rv)* {(I r (J'R) 4, 0} {S4, (jk+c ')} {Rp I(Rp)*} {(I rG'J?)4 0} 1 F.(1). (*301L02). *301-12. *300-14.)D F: S 4,r(L+ 1) E nurn (IR).)D. S4,(+c 1) e Rp"cntm (R). a! p~+1. [*301'1] D) (aP, v). P4,v c numn (R). S 4, (,a ~01) = (P IA) 4, (v +0 1). a! g +1. [*55-202. *120-31 1] D. (HP). P4, tnUM (R). S4 (+01) =(P IR)4, -f0)(2) F.(2).) F:. P p4, Q 4, j4enurn (1?). p, Q. P = Q:)D F.(3). *301-12-13. Induct. D F. Prop *301-141. F. G'h'ntimn(R)= C'Un~t3'lR Dem. F.*301-1.:) F aoe 1 U'U it3'R.oeGN'num (R))(ao-~01),6CU'hlnum (R) (1) F.(1). *300'14. Induct.:) F. Prop

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248 QUANTITY [PART VI *301-15. F. 9'num (R) e 1 — +) Cis Dem. F.(1). *72-32.) F. Prop *301416. F:pE C'Urit3'R.=-.E I 1'A [*301-141-15.(*301103)] *301-2. F-. I lG'R. Ri=1? [*301-13-161. (*301-03)] *301-201. F:veC'Unt3'R.).(R~4v)enum (R) Dem. F - *301-16. (*301-03). D F: Hp. D. R,' t9'num (R)J v. [*41-11]. (HM). ME nim (i). RMv. [*301-12]. (gMI, P, a-). M enu m (R). M =P 4 a-. Rk'Mv. [*55-13].(RI' Iv)enum (R): D F. Prop *301-21. F:vJJI'Urit3'R.:).Rvl+cl=RvIR Dem. F. *301-1201. DF: Hp.D. Rv+cl4(v +,a 1), (Rv IR) (v~0,,1) enum (R). [*301-14]:). Rv+cl =Rlv IB::) F. Prop *301-22. F: E!Rv.).RvePotid"R [*301-201-12-16] *301-23. F:uvCU t"..g-"RlvRII [*301-21. Induct] *301-24. F:.o-eNCind:,u ~rG.V<).kD, v.-RAtRv:). P~ t(g'). p <,' ~r. P =R'~}eca-+.l1 Dem. F.*120-442.)F:Hp.,u < v a-.1RP= Rv.:). uv (1) F.-(1).*7 314. *301-15. FHp. D.Nc'lPt(ap) a-.P = Rl-=Nc6' (p ~a-) (2) F. (2). *1 20-57.)F.Prop *301-241. F: Hp *301P24.:). a- A t2rE P(lU ~ t3'fR). RT-+cl R= I [*301-24-104-21] *301-242. F~-CUtlR,,,-v,. R=vDRI=o-!Lv Dem. F. *1 20-412 -416.:) F: Hp.). a = (a- cP)+ ~ [*301-23] Ro = ~ Ra- IRt [Hp.*301P21] Ro I R = RacL BI R+cl [*301P23] = Ra-cP+cv+cl::) F. Prop

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SECTION A] NUMERICALLY DEFINED POWERS OF RELATIONS24 249 *301-25. F: (Ho-).P= RO'.).(H).P IR = RK [*301-16-241,242] *301-26. F:P ePotid'R.=. (:qo') P =R Dern. F.*301P25-2.Jnduct.:) F: Pe Potid,'R.). (a-). P = (1) F. (1). *301-22.) F. Prop *301-3. F:RC-I.ueC'Urit3'R.:).Ry=R=Ro=IrC'R [*300-312. *301L16-26] *301-31. F:RC1I.o-tO.:).R,,=A [*-300-48] The above proposition is the same as *300-48, but is repeated here to show the relations of R,. and RK. *301-4. F: ReRelnum.a-EC'Unit3,R.:).R,7,=Kl Dem. F. *301P2. *12130 2.) F: Hp.).Ro=1R (1) F. *301P21. *121-332.)D F.(1).(2). Ind uct.)F.Prop *301-41. F e m.ft!Ri I cc [*30P4-16. *300-55] *301-5. F "UnVR 4+0.v R11 ~x~ Dem. F.-(1).*301-16-2:) F: Hp.).(Rix)'l=RKxcl (2) F.-*301-23. D i 3R (ix,, Rk, t (3) F. (3).-*301P23.)D F:(,t X0 V) +, EC' U rn t'R. (RIA)v = RLxcv. D. (Riu)v+cl = RGtLxcv)+cfk (4) F.-(4).-*113-671.)D F:.(RA)v — LRCV x) v. D x0 v+, 1 C UA nt3R.).(RA)Y,+cl =RIAxc(v~cl) (5) F.- *1 17-571P32 F: ~ X0 (V+v+1)EC'CCU nt3'R..fa Xc VE6C'U A t3'R (6) F. (5).(6).:) F:. 4 x. v CE CU ri t3'1R.).(Rfk~ -Rtp~cv. pu X 0 (V. +0, 1) e 1 GUAn t3'l. R (RIA)v+cl = R'hxe(v+cl) (7) F.-(1).(2). (7).J nduct.)D F. Prop *301-501. F: =a 0. ve C' U rn t'R. (R*L)v -RBlxcV [*301P2.3] *301-502. F,tvC1U n t21CR. X). V1a x0veCU t,'R.(,a XC V) At2'Be C'U Dem. F.*300.14. *120-5.)DFHp. a! (/x"V) rit2"R.)D. (p xv) rit2,,R eGU (1) [*113-17.*64-61]).(gaa,). a x 3 e (,a x, V) A t2', R (2) F.(1).(2).*65,13. DF.Prop

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250 QUANTITY [PART VI *301-503. F v eNCi md. vn tP'6R EC'U~(t2'U',R).:). ve ECG'C(U~ t3'?) Dem. F.*300-14.)F: Hp.:)(a.,vt,,R [*106-2]:). (axr, a).4. x,',aE ii n t2'R(1 F.(1).*300-14.)F.Prop *301-504. F:,O,ye C'Un t2'G'IR. v40.).(-RI)v,=R/ LXcP [*301IA5-501 -502'503] *301-505.F::DU)! (4) (U}:XVECU Dem. F.*120-452.)

F:!(e4)GU}". A~(C)auO.EU F. (1).-*300502. *301-4.) [*300-572].4x,,v eG'U`:. F. Prop *301'51. FC.:"D'. ~ f —4) GUJVA {(C',) CU}'J. 4:X0 V E G"U. 4:x0 v = X0 fL Dem. F. *301'505.*300-232. *301-4.) [*300-571] x4:XcveGC'U. 4:xcv= x0, u F. Prop *301-52. F:veR EDUA% t'R. (x0, p)v x0,(,a") Derm. F. *301-2. *1 13-204. *1 16-204-321.)F. (x. X. (,U) (1) F.*301-21 F F: VE 'cJ'Unt'R.:).(X FL)v c(xoO)JI (xC (2) [*1 16-52-321] = X0(vcl) (3) F. 1.(3). Induct. F F. Prop

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*302. ON RELATIVE PRIMES. Summary of *302. The present number is merely preparatory for the definition and discussion of ratios. We want, of course, to give a definition of ratio which shall ensure that ju/v = (, xoT)/(v x r). Hence in defining tY/v in any given type, we cannot exact that p and v themselves should exist in that type, but only that, if p/a is the same ratio in its lowest terms, p and a should exist in that type. Hence, if we are not to assume the axiom of infinity, it is necessary to deal with relative primes before defining ratios. The theory of relative primes is concerned with typically indefinite inductive cardinals (NC ind). It will be observed that we have three different sorts of inductive cardinals, namely NC ind, NC induct, and C'U. NC ind consists of typically indefinite cardinals, NC induct consists of all cardinals of some one type, and C'U consists of all existent cardinals of some one type. If the axiom of infinity holds, we have C'U= NCinduct, and NC ind = sm"NC induct. But neither of these is true if the axiom of infinity does not hold. It will be found that, where we require typically definite cardinals, it is C'U or (I' U or D'U that we require, not NC induct; that is to say, we almost always want to exclude A, and sometimes we want to exclude the greatest existent cardinal of the type in question, or to exclude 0. Thus "NC induct" will seldom occur in what follows. The cases in which C'U or D'U or a' U occurs are of two sorts: (1) where we are proving typically definite existent-theorems, (2) where we are concerned with series, as e.g. in *300, where we considered the series of existent cardinals, or in *304 below, where we shall consider the series of ratios. Wherever series are concerned, we must have typical definiteness, because the definition of "PeSer" involves C'P, and therefore only a homogeneous relation can be serial. This is a particular instance of the fact that when we require numbers as apparent variables (as e.g. in the theory of real numbers), typical definiteness becomes essential. Many propositions containing the hypothesis "e NC ind" (where p is a real variable) do not allow of p being turned into an apparent variable, because this requires that,p should be fixed in one type, and our original proposition may demand that the

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252 QUANTITY [PART VI type in which, is fixed should be a function of A. For example, *300*17 states F: e NC ind. D. (ga). t e C'(U t2'a). If we fix the type of A, we thereby also fix the type of a, and the proposition becomes false unless the axiom of infinity is true. In fact, the proposition demands that, the greater p

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becomes, the higher must the type of a become. " NC ind " is not a strictly correct idea, and the primitive proposition *9'13 does not apply without reserve to propositions in which it occurs. We have introduced it because it immensely simplifies the expression of many propositions, and because it is easy to avoid the errors to which it might give rise if used without remembering that it is a concession to convenience. It will be found that, when we are not concerned with existence-theorems, or with numbers as apparent variables, " NC ind " is almost always the notion required. This applies to all cases where we are only concerned with addition, multiplication, subtraction and division; it applies to ratios except when ratios are considered as forming a series, or when we are investigating their existence. As regards the use of an "NCind" as an apparent variable, there is a distinction between "all values" and "some value." If we have "p e NC ind," "(gp)" will often be legitimate when "(p)" is not. The reason of this is that, if we are to fix upon one typically indefinite cardinal, it will be possible to assign one definite type in which it exists; e.g. there are at least two classes, four classes of classes, sixteen classes of classes of classes, and so on. But if we are making a statement about all typically indefinite inductive cardinals, it will not be true unless there is a type such that our statement holds of all inductive cardinals in this type. In virtue of *300-17, if we have "pe NC ind," we may replace it by "peC' U" if we may take U in as high a type as we please, or if, on account of the rest of our proposition, p cannot be greater than some assigned inductive cardinal so long as the hypothesis of our proposition is true. The above remarks will enable the reader to test the uses of typically indefinite inductive cardinals as apparent variables, and the passage from propositions concerning NCind to propositions concerning C'U. We define p as prime to a- when both are typically indefinite cardinals and 1 is their only common factor, i.e. we put *302'01. Prm = pa {p, a e NC ind: p = x x. a = xo. t,,,r. = 1} Df In this definition, r, v, T may be taken to be typically indefinite cardinals, because, when p= XoT. <r0= Xo T, we must have 4 6 p. v a<. rT p. rP a, so that 4, a, T cannot grow indefinitely (with a given p and a) while the hypothesis remains true. We define "(p, a-) Prm, (/A, v) " as meaning that p is prime to a-, that r is not zero, and u = p x r. v = oa xo T, i.e. p/ a is /t/v in its lowest terms, and T is the highest common factor of / and v. The definition is:

 *302-02. (p, a-) Prm, (i, v). = p Prm -. T e NC ind - t'0. = p x, v. v = oa Xo Df We then put further *302-03. (p, a) Prm (u, v). =. ([Tr). (p, -r) Prm, (p, v) Df Here again there is no objection to r as an apparent variable, because T must be not greater than p and v. "(p, a) Prm (,a, v)" secures that p/a is /lv in its lowest terms. We also define, in this number, the lowest common multiple and the highest common factor. Our definition of "Prm" is so framed that every inductive cardinal is prime to 1 (*302'12), that 1 is the only number which is prime to itself (*302'13), and the only number which is prime to 0 (*302'14). After a number of preliminary propositions, we arrive at the result that if u and v are not both zero, and 4 and q are not both zero, the existence of a couple p, o- which is prime both to p, v and to A, vR is equivalent

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to p Xo, r= v xo, i.e. *302'34.:., v,, e NC ind. (= = 0). (= = 0).: x, X0 7 = v X0.. (p, o a) (p, -)) Prm (,a, v). (p, a) Prm (~, n) We have also *302'36. F: p, v e NC ind. (,a = v = 0). -. (3P, a). (p, o) Prm (,a, v) *302-38. A: (p, r) Prm (p, v). (~, a) Prm (, v). D. p =. a = 77 I.e. there is only one way of reducing a fraction to its lowest terms. We prove also (*302-15) that if p, v are typically indefinite cardinals, which both exist in the type of X (i.e. px, v, e C' U), then (p, a) Prm (pu, v)..(p, a) Prm (,A, VA). This enables us, when we wish, to substitute typically definite cardinals for the typically indefinite,p and v. *302-01. Prm= p {p, o- NC ind: = x r. = x..= 1} Df *302-02. (p, -) Prm, (Hp, v). =. p Prm a. - e NC ind - 0. = p x v. v = a- x Df Here pa, v are to be typically indefinite in the same way as p xc Tand c x Tr. Thus if, in some one type, p xc v and ar xo 7 are both null, that does not justify us in writing (p, a) Prm, (A, A), because there are other types in which p xc T and oa x T are not null. On this subject, cf. *126. *302'03. (p, a) Prm (it, v). =. (gr). (p, a) Prm, (,u, v) Df *302'04. hcf (,u, v) = (?) {(p, a). (p, a-) Prm, (p, v)} Df *302-05. 1cm (4, V) = (?:) {(gp, aT, r). (p, a) Prm, (H., v). = p xc 0 x, T} Df a302~05. Icm(~~~~~~~~~~~~~~~~~~~~~~~~~,v) Df~~~~~~~~~

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254 QUANTITY [PART VI *302-1. F:.pPrmo.=-:p,o-e NCind:p= x,-. a = i7 xT. T,,,. = [(*302-01)] *302-11. F: p Prm.a-. =. o-Prm p [*302-1] *302412. I:pPrml.=. pENCind [*302-1.*117-631P61] *302-13. F: p Prmp.. -. p=I Dem. F. *302-12. F: p=1.D. pPrmp (1) F.*302-1. DF:.pPrmp.:p=1xr, p.). p=: [*113-621]):p=I (2) *302-14. F:0Prm.t=. =1 Dem. F-.*30212. D:= 1D. 0.Prm, (1) F.*~302~I1. D ~:. OPrm/.k. D::O =Oxr tL. p==1 x. A.:). p ~= I:: [*113-601V621] Du = 1 (2) F.(I). (2). D F..Prop *302-15. Fz, v e NC ind. PA, VA E C' U.: (p, a-) Prin (/L, v)..(p, a-) Prni (,uk, vA) Dem. F.*126-101.*300 14.) F:. Hp). p Prmn a- * T e NC ind - t'0., = pX r T. = a- *T. p Prma-. r e NC ind - t'0.., = p X0T. vk = a-x (1) F. (1). (*30202-03). D F. Prop *302-2. F:,a, VE O U. V = 0). = T^ (al, a). 1L = P X0 T. V = a- X0 T1. D E! max (U)'Ic. max (U)'xl e D'U Dem. F. *113621.)F: Hp.). 1 Ic (1) F. *117-62. *113-602. Transp.) F:. Hp. Te ic.:T,a.v.-T<v (2) F. (1). (2). *300-21-22. *2626. *300-26.)F. Prop In the above proposition we write " max (U)'ic" rather than " min (U)'x," because, since U arranges the natural numbers in descending order, "min (U)'xI" is the greatest number which is a member of Ic, and therefore it is less confusing to speak of this nu ruber as " max (U)'Ic."

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SECTION A] SECTION A] ON RELATIVE PRIMES25 255 *302-21. F: Hp *302-2. Tr= max (U)'K., 1i=p XT. V = U X0 T.) Dem. ~ ~~~~~~~~~~~~~~~~(p, a-) PrinT (At, v) F. *13-12.)F: Hp. p = p' X0 T'. 0- = 0-' X0'-T'.) 1 =p' X0 T'X07T. V =GTX0,T'X07. [*113-602.Transp.Hp]:). T'x, x7'-#.,r'X TX< T. [*120-511.*117-62]). T'=1I (1) F. (1). *302-1. DF: Hp. D. pPrm a (2) F. (2). *302-2.(*302-02).:) F. Prop *302-22. F:.,a, v eNCind (Atb= v= 0).D: (ap, a,T).(p, a) PrMT (A V): (2[p,

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a-). (p, a-) Prm (At, v) [*302-2-21. *300-17. (*302-03)] *302-23. F:.I.,vcD'U.:): (Ujp,o-):p,o-,ED'U.Atx(,o-=vx~p: 4:, 77cD'fU. At x01 = v x0 4:.)I", 4:,>p. -n 0 - Dem. F. *300-23. *1 13-27. F. (1).*300K12.)D F:. Hp.): (appa-):pa-cD'U.pAtoo - =vxcp 4:,neD'U.Atxc0q=vx04.:.)t,.4:>p (2) F. *120-51.) F:Hp. p,o-,cD'U.Atx(o- = PXCP-P.AxCy7= I' xC4:)P xflq xC4 (3) F.*11751.)F:Hp3).4:,nD'U.4:p)4x(,a-,p x0a- (4) F.*126-51.)F:llp(4).a->nq.).px~a->px0'J (5 [Transp])4X. a- = p x0q.)-, q> (6) F. (2). (3). (6).)F. Prop *302-24. F:.Ai, v, p,(TrENC Ind - t'O.AtOxca-=lX p: At X q = v x0 4:. 4: 6 cDU.)~4 >p a-:).p Prm aDem. F.*302-1.)D F:p, a-c D' U. e, (p Prm a-). D ~: q 7-). -r 4 P 1:X0 T. a- =q iO [*113-203-602.*120-511.*1 17-62]).(a4:?,vr).:,nTc'U-t'.4:<p~~a-.p=:x07.a=nx~r(1) F. *120-51. D F:,at, V, p, ae ED' U. At x0 a- = ii X,, P. P=4:X,, 7 T = fl XO 7.) PtXc f =I'X04 (2) F. (1). (2).) D F:pAt v, p, a- e D' U. At xO a- = v x0, p. (p Prm a-).) (~4:,n.Atx~~vx04:4:, nc'U.4:<~n~a-(3) F. (3). Transp. *300-17.) F. Prop

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 *302-25. F:p,~ED'U.:).(gja,/3).aEG'U.13<~.p=(ax0~) ~,0/3 Dem. F.*117-62.*120-428.:)FI:Hp.).p <(p-+-0l)x0. (1) -. (1). *300-23.)F: Hp.). E! min(U)'& aae C' U. p <(a ~01) x0 J. [*120-414-416]. (E~a).aE C'U. p< (a~01)x0..p >a x0, [*117-31.*120-452]).3,3.,30Up(+1 0. p =(a x04)-i-0/. [*120-442.*1 17-561.Transp] * aa /) ae (J'U. /3 < ~. p =(a x0, +0/3::) F. Prop *302'26. F:Hp *302-24.).(p, a-) Prm,u v) Dem. F. *302-25. D F.*113-43. D F:, = (a x0 p) +0/30. v = (ry xra-) ~0r8. /3 < p. 8 < a-. /. X a- = v x0 P.) (axcpxca-)+0 (/3xca-)=(,yxop xoa)+0(8 x~p)./3< p.8 <a-. (2) [*1 17-31.*120-452.*113-671]:). ax~ px>%- <(7+01) x0 xoca-. y x0p x a- <(a +01.)x0op x a-. [*126-51]:).a<ry-I-.ry<a~01. [*120-429-442.*117 25"]:).a = y (3) F.(2).(3).*120-41.)F: Hp (2).). /3x0a-=8 x0p. /3< p. 8<a-. [Hp]:)./3==0.8=0 (4) F.(3).(4).:)F:Hp (2).:).,tt=axop.v=ax~ra- (5) F.(1). (5). *302-24. DF. Prop *302-27. F:p, v, p,a-4y~,e NC ind - t'0.,x, a = vx P 1x0n=v x04: Dem. F. *113-27.) F: Hp.. xc xv x0, a- = 77 x01t xe a[Hp] = Iqx v XeP. [*126-41] x). 4xa=, x,,p: D F..Prop *302-28. F: Hp *302-24. 77qeNC ind - t'0. p xcq = vx04.:.). (p, a-) Prm (4:, n) [*302-26-27. *300-17] *302-29. F:Hp*302-28.4:Prmq.:).4:=p.nq=oDem. F. *302-28-1) F:. Hp.): (aa).:= a x~p.q= a xc-: 4:= a xcp = a Xo-.)a,. a =1: [*14-122] D:4: =1x~p.?7=lx,,o-:.)F.Prop

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 SECTION A] *302-3. F: /, v,:,flENC ind - tf0. /ix07)q= v x04:.). (:ffp, a-). (p, a-) Prm (p~, v). (p, a-) Prm(4,) Dem. F-. *302-23-24.) F:. Hp.:): (up,o-): p Prmoa-. p, a-eNC ind- ff0.1,t x,,- = v x(,p: a, /3 E D'" U. u x0 /,3 =v x, a.),. a >, p.8 /3 -: [*302-26-28]): (:p,cr). (p,a-) Prm (p, v). (p,a-) Prm(4:, i):. F.Prop *302-31. F:(p,o-)Prm(~,v,)..LtPrmv.:).FL=p. v=aDem. F. *302,1. (*302-02-03).) F:.Hp.:):(H-) -/= p X,,r. V= a- X0 r:,z pc xT. ~*V=0 ac T~. D,. T=l [*14-122]:).,O=p x01 I. v =a- x01,:.:) F.Prop *302-32. F:4: Prmn.ptPrmv.4:x,,v=nqx0F..4=.n Dem. F. *302-3-31h F:IHp.). (p,oa-). pPrmoa-.:

http://quod.lib.umich.edu/cgi/t/text/text-idx?c...stmath;rgn=main;view=text;idno=AAT3201.0003.001 (130 of 246) [5/26/2008 7:31:50 PM]

= p. = p. 7=a-.*v = -: F. Prop *302-33. F:.kt,v,4:,7)cNCind-t'0.D: P xe 77 = ii x0: O (Hjp, a-). (p, a-) Prm (p~, P). (p, a-) Prm(4,) Dern. F. Id. (*302 -02 -03). D F: (p, a-) Prm (j.z, P). (p, a-) Prm (:)) (ar, 7'). T, 7,E D"'U. / = (p Xe T. V) a- X0 7. 4=p X0, T7.77 (T a- T~ [*113-27]).(aT,7T').,IIXC7) = pxcX O-x,07X'T =X04(1 F. (1). *302-3. D F. Prop *302-34.F:.,,,7 Nin.Q==)c(=7 0): Dem. F.*113-602.)F: Hp./-=0.v+0.).4:=0.n#0 (1 F. *1 13-602-621.) [*302-14] D. (0, 1) Prm (I., v). (0, 1) Prm (4,) (2) F. (1). (2).:) F: Hp. = 0. -v + 0.) Similarly F: Hp av 0.p+ 0. (ajp, a-). (p, a-) Prm (,at, v). (p, a-) Prm (4,) (4) F. (3). (4). *302-33.)F.Prop *302-35. F:./.,veNCind.-~(pz=v=0).pPrma-.): x~a-=vx~p.- -.(p,oa-)Prm (u, v) [*302,34-1431] R.& W. III. 17

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2.58 QUANTITY [PART VI *302-36. F~~e~n.-,,~=)=.H~-.po)r(uv Dem. F. *302-14.:) F:. (p, o-)Prm (p~, v).: a, - e NC ind.-, (p = a- = 0): (p-).r c NC ind - tf0. tt= p x0, -r.v = a- x, -r: [*120-5.*113-602])~z Cm ~=v=0 1 F. (1). *302-22.:) F. Prop *302-37. F (p, a-) Prm (Ii, v). I,v,vENCifld.'%.'(,L=0.v=0).pPrma-./Itx,,-=vx,,p [*302-35-36] Dem. F.*302-37.)F:Hp.). pPrmoa-. ~Prm'q. ux a- = v x0cp, P = v x0 ~. '-(Q=.LOV=0) (1) F. (1). *302-14.*113i602. DF: Hp. = 0.D. p =0.4= 0.oa-=I.fl = 1 (2) F.(1).*302-14.*113-602.)F:Hp.v=0.).p=1.4=1.a-=0.nq= (3) F. *302-27.)F:Hp. p [. v = 0.).P x,,70 =a- x0 F. (2). (3). (4).) F. Prop *302-39. F: (p,ao) Prm (p, v).).u p. v - Dem. F. *302-23-36. D F:.,au, y e D' U.) (ap, a-): (p, a-) Prm (g v): ~,q D'Uqx. q = vx0, p.)~.q a -: F.*302-37-14. DF:,a=0. (p,ao-) Prm (/.,v).). v 0. p= 0.O- =1 (2) Similarly F v =0. (p,o-) Prm (/.,v).).p~u 0. p=1I.o- = 0 (3) F. (1).(4). *302-36. *300-17.)F. Prop *302-4. F:,ve NC ind (,a v=0))E!hcf(1tv) Dem. F.*302 -2 2. Hp.).(ap, o-,T). o-) PrmT~, (v)(1 F. *302-38. (*302-02-03).) [*126-41]:).Tr=w (2) F. (1). (2). (*302-04).:) F. Prop [Proof as in *302-4]

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SECTION A] SECTION A] ON RELATIVE PRIMES25 259 *302-42. F:p~vENCinid., (~=v=0)).D.hcf(jt,v)x01Icm(i,v)=x,uxv Dem. F. *302-441. (*302-04-05).) F:Hp.) D. (pa-, r). p= p x0, '- v = a x- 7-. hcf (FL, V)= T. 1CM (,t, V) = Xa-X(T hcf (/, V) X,1 CM (,t, V) = pX,, 0- X. T2::) F. Prop *302-43. F:(p,uo) Prm (z, v). D.p x.hcf (p, v)=.o-x,,hcf (p,v) v [*302-4. (*302-02-04)] *302-44. F:(p, o-) Prm (p,v). D.p x,,v =1cm (j,uv) = a x0 [*302-41. (*302-02-05)] *302-45. F:(p, o-) Prm(F,av).,flNC ind c.(=l0).F x,,flq x04. 1cm(~ P Xp x*,= a Dem. F.*302-3Z)F:Hp)(p,o-) Prm (~, i)(1 F.(1). *302-44. DF. Prop 17-2

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*303. RATIOS. Summary of *303. In this number, we give the definition and elementary properties of the ratio pL/v. Most of the important applications of ratios are to numerical or identical relations, i.e. to relations which may, in a certain sense, be called vectors. Neglecting identical relations for the moment, let us consider numerical relations, and to fix our ideas, let us take distances on a line. A distance on a line is a one-one relation whose converse domain (and its domain too) is the whole line. If we call two such distances R and S, we may say that they have the ratio p/v if, starting from some point x, v repetitions of R take us to the same point y as we reach by p repetitions of S, i.e. if xR^y. xSly. Thus R and S will have the ratio pJlv if a! R nA So. In order, however, to insure that /v-V = p/a if (p, o-) Prm (pL, i), it is necessary, in general, to substitute a! RB n SP for X[! RB A So. (In the above case of distances on a line, the two are equivalent.) Thus we shall say that R has the ratio Ft/v to S if (Hp, a). (p, o) Prm (ip, v). [! R n/ SP. If we apply the above definition to identical relations, we find that, if R C. S C I, R has the ratio FL/v to S provided 3!R S, i.e. provided [! C'R n C'S. This application is required for dealing with zero quantities and zero ratios. Thus we are led to the following definition of ratios: *303 01. FL/v = RS {(gp, a). (p, a) Prm (/p, v). [! R!BA SP} Df This definition, as it stands, requires justification in two respects: (1) we commonly think of ratios as applying to magnitudes other than relations, (2) we should not commonly include as examples of ratio certain cases included in the above definition. These two points must now be considered. (1) In applying our theory to (say) the ratio of two masses, we note that the idea of quantity (say, of mass) in any usage depends upon a comparison of different quantities. The "vector quantity" R, which relates a quantity ml with a quantity m2, is the relation arising from the existence of some definite physical process of addition by which a body of mass ml will be transformed into another body of mass m2. Thus a- such steps, symbolized by RB,

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 represents the addition of the mass - (m2 - ml). Similarly if S is the relation between M2 and M1 which arises from the process of addition turning a body of mass M1 into another body of mass M2, then SP symbolizes the addition of the mass p (M, - M1). Now X! Ro i SP means that there are a pair of masses m and m', such that mRir)Z' and rnSPm'. In other words, if we take a body A of mass m and transform it so as to turn it into another of mass m4 + - (m, - mi), we obtain a body of the same mass m' as if we proceeded to transform A into a body of mass m +p(M2-M1). Hence a (mn - mi) = p (M2 - M1); that is (m2 - m)/(M2- M)= p/cr. But in our symbolism the addition of m2 - ml is represented by the vector quantity R, and that of M2-M1 by the vector quantity S; so in our symbolism R has to S the ratio of p to o. Thus to say that an entity possesses,u units of quantity means that, taking U to represent the unit vector quantity, U* relates the zero of quantitywhatever that may mean in reference to that kind of quantity-with the quantity possessed by that entity. It can be claimed for this method of symbolizing the ideas of quantity (a) that it is always a possible method of procedure whatever view be taken of it as a representation of first principles, and (/3) that it directly represents the principle "No quantity of any kind without a comparison of different quantities of that kind." Furthermore analogously to our treatment of cardinal and ordinal numbers, we can define any definite quantity of some kind, say any definite quantity of mass, as being merely the class of all "bodies of equal mass" with some given body. The zero mass will be the class of all bodies of zero mass; and if there are no bodies with the properties that a body of zero mass should have, this class reduces to A in the appropriate type. Thus the application of our symbolism to concrete cases demands the existence of a determinate test of " equality of quantity " of different entities, and a determinate process of " addition of quantity." The formal properties which the process of addition must possess are discussed in the numbers concerned with vector families. (2) Having now shown that cases apparently excluded by our definition of ratio can be included, we have to show that no harm is done by our inclusion of cases which would naturally be excluded. In order that ratio may agree with our expectations it is necessary that the two relations R and S, whose ratio we are considering, should have the same converse domaifi. Otherwise we get such cases as the following: Let P, Q be two series, and suppose* B'P=B'Q, 5p=6Q, 11p=9Q, 13p=25Q, but that P and Q have no other terms in common. Then we shall have, if R = P. S = Q, (B'P) R45p. (B'P) S5 5p, * For notation, cf. *121.

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 whence it follows that R has to S the ratio 5/4, i.e. we have R (5/4)S. But we shall also have R (8/10) S and R (24/12) S, i.e. R (4/5) S and R (2/1) S. Thus our definition does not make different ratios incompatible. In practical applications, however, when R and S are confined to one vector-family, different ratios do become incompatible, as will be proved at the beginning of Section C. And so long as we are not concerned with the applications which constitute measurement, the important thing about our definition of ratio is that it should yield the usual arithmetical properties, in particular the fundamental property p1jv = p/a.-. Xc C = V Xa P, which is proved, with our definition, in *303-39. Thus any further restriction in the definition would constitute an unnecessary complication. In virtue of our definition of Fp/v, p/v = A if p and v are not both inductive cardinals, or if = = v = 0 (*303-1114). We have (*303-13) }. t/v = Cnv'(v/,u), i.e. the converse of a ratio is its reciprocal. If p = 0, and R (p/v) S, R must have a part in common with identity (which we may express by saying that R is a zero vector), and S may be any numerical or identical relation whose field has a member which has the relation R to itself (*303'15). Thus if v, a are inductive cardinals other than 0, 0/v = 0/a. The common value of ratios whose numerator is 0 is the zero ratio, which we call Oq (where "q" is intended to suggest "quantity"). The definition of 0q is *303 02. Oq = '0/"NC induct Df In like manner, if Ft and p are inductive cardinals other than 0, we have Ft/O = p/O. The common value of such ratios we call oo q, putting *303 03. oo q = s'/O"NC induct Df The properties of ratios require various existence-theorems, and in establishing existence-theorems without assuming the axiom of infinity, the question of types requires considerable care. We have *303-211. F: (p, ao) Prm (p, v). D. p/v = p/a so that the existence of p/r does not depend upon pu and v, but upon p and a, where p/a is pL/v in its lowest terms. We may, therefore, in considering existence-theorems, confine ourselves, in the first instance, to prime ratios. To prove the existence of (p/a) t'R, when p Prm a, we take two relations R and S both contained in identity. These have the ratio p/a- provided their fields have a member in common and E! RB. E! S. By *301*16, this requires p, - e C(U t3'R). Thus we have *303'25. F:. p Prm.)::! (p/la) t'R. - p, a- e C'(U t3'R). -. p (R), a (R) e C'U

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SECTION A] RATIOS 263 But this existence-theorem, which is obtained by supposing R and S contained in identity, is not much use in practice: what we require is the existence of a ratio between numerical relations. For this purpose, assuming p > a and a- # 0, let X be a class of such a type that Nc't'X > p +c 1. (Such a class can always be found in some type, by *300'18.) Then we have PA e ('U, and we can construct a series Q such that C'Q is of the same type as X and Nc'C'Q = p +c 1. (This is proved in *262-211.) We can then choose out of Q a series P having the same beginning and end, and consisting of a- +o 1 terms. We then have (B'Q) (Q,)P (B'Q). (B'Q) (P), (B'Q). Hence Pi and Q, have the ratio p/a. A similar argument applies if a p and p = 0. Thus we arrive at the proposition: *303-322. F: p Prm a. pi, ao e D' Un (I U. D.! (p/a) t (Rel num n t'X) I.e. if p is prime to a and neither is 0, and p +c 1, ao +o 1 both exist in the type of X, then there are numerical relations having the ratio p/a and having their fields of the same type as X. The case when either p or a is 0 requires separate treatment. If R has to S the ratio 0/a, R must be partly contained in identity (*303-15); hence we have to find a hypothesis for X! (0/a) r Rel num, since 3! (0/a-) Rel num is impossible. Since 0/- = 0/1, we only require the existence of 2 in the appropriate type, i.e. we have,303 63. F: [! 2A. 3.! Oq (Rel num n to'X) It will be remembered that g! 2, is demonstrable except in the lowest type. In the above propositions, pu and v and p and a- have been typically indefinite. Ratios of typically definite inductive cardinals are dealt with by means of *302'15, which gives at once *303-27. F::,, v e NC ind.,a, v e C' U. D.,//v = A!/v I.e. a ratio may, without changing its value, have its numerator and denominator specified as belonging to any type in which both exist. This enables us to take p and a- as typically definite cardinals in *303'322, thus obtaining the proposition *303'332. F:. p Prm a-.:: [! (p/a-) ((Rel num n tl'p). p, a e D' Un (U The above existence-theorems are useful in proving a//3 = ay/8. -. a XC 8 = x, y. We proceed as follows: We first show (*303-34) that, if p, a are inductive cardinals other than 0, and p +c 1, a +o 1 exist in the type of X, we can find numerical relations R and S such that X! Re SP, but q > a-. D. X[! R'.

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264 QUANTITY [PART VI This is done by taking two series P and Q having the same beginning and end, and having C'P e a +c 1.C'Q e p +c1. Then if R = P, and S = Q1, we have (B'P) Ra (B'P). (B'P) SP (B'P): v > a. D. Rn = A, whence the

result. From this proposition it follows immediately that if p Prm o. Prm.. > o-, and if PA, a-(D'Un a'U, we can find an R and an S such that R (p/o) S., {IR (4/n) S}. A similar argument applies if 77 < a or > p or $ < p. Hence we find, by transposition, *303-341. F: pA, o- e D' UN (I U. p Prm a-. Prm 97. (p/-) CtooX = (/7) too. D. p=. *-= v From this point on, the argument offers no difficulty. For if we have a/ft = 7/. (p, a) Prm (a,,8). (I,) Prm (7, 8), we have, by *303-341-211, p =. a ==. Hence, by *302-32, we have a xc 8 = i x 7r. What is approximately the converse, i.e. *303 23. HF: t, v,, rl eNC ind. (= = 0). (f= 9 = 0). a x0 =) X -. 7 =.). JV = /X7 follows at once from *303'211 and *302-3. Hence, after dealing with special cases, we find *303-38.:.a,/,,7,8ENCind: aA, A e a' U. V. Y, A (I' U: (a = = 0). (7 = 0):: (a/LP) too'X = (Y/): tooX0 -. a x, 8 = pl x ry It will be observed that a//I is typically indefinite, like Nc'~. But in order to insure that a/fl = y/S however the type may be determined, it is only necessary to insure that this equation holds in a type in which (a/,/) Rel num exists. When we write simply " a// = y/8," we shall mean that this equation holds however the type may be determined; in other words, that it holds in a type in which (a/,I) Rel num exists. (There always is such a type, if a, 8 e NC ind - t'0, in virtue of *303-322 and *300-18.) Thus we have *303-391. F:.a,/ e NCind. aA, f^ e (I'U. -(a = = 0).:): (a/I) to0'\ = (r/8) to'X. -. a/I = /8. -. a x, 8 = p x, 7 and, in virtue of *303-38, we have *303 39. F:. a, t, 7, r e NC ind.,-(a = - = 0). (y = 3 = 0). D: a/I3 = y/. -. a xo s = p x, This proposition is, of course, essential to the justification of our definition of ratios. The remaining propositions of *303 consist (1) of applications of the theory of ratio to powers of a given numerical relation, (2) of properties of Oq and oo q, (3) of a few properties of the class of ratios. This last set of propositions depends upon two new definitions, which must be briefly explained.

 We have already explained that,L/v is typically indefinite. Thus if we call /Iv a "ratio," ratios are, like "NC ind," not strictly a class, because every class must be confined within some one type. Nevertheless it is convenient, just as in the case of NC ind, to treat ratios as if they formed a class; and, with similar precautions, we can avoid the errors into which we might be led by treating them as a proper class. We therefore put *303-04. Rat = X {(I[p,)., v e NC ind. v + 0. X = /v} Df (The condition v 4 0 is only introduced because it is usually convenient to exclude oo q.) It will be observed that pL/v is still typically indefinite if p and v are typically definite. This results from *303'27. But we often want typically definite ratios. We want these defined in types in which there are numerical relations having the ratios in question. Hence we put *303 05. Rat def= X {(lu,v). p, v) e D' Un (' U. X = (/I) tll',} Df Here "def" stands for "definite," and pA, v are typically definite inductive cardinals. The desired properties of "Rat def" result from *303'322. It should be observed that, besides consisting of typically definite ratios, "Rat def" differs from "Rat" by the exclusion of Oq. This is merely for reasons of convenience. The properties of " Rat" and " Rat def" follow immediately from previous propositions. We have *303 721.: X e Rat - t'Oq. D. (a {). X: tllp e Rat def *303 73. F: X e Rat def. D. [! X Relnum By *303-322; and by *303-391, *303'76. F:.X, YeRat. X t'p e Ratdef.: X tl'p = Y tl'p.. X = Y If the axiom of infinity holds, every member of " Rat" except Oq becomes a member of " Rat def" as soon as it is made typically definite. Hence *303'78. F: Infin ax.. Rat def= Rat - t'Oq The uses of " Rat" and " Rat def" differ just as the uses of " NC ind " and "NC induct" differ. The distinction is only important so long as the axiom of infinity is not assumed. *30301. p/lv = RS {([p, r). (p, a) Prm (,u, v). [! Re n SP} Df In the above definition, p, a, pt, v are typically ambiguous, but p, a- must (by *301'16) exist in the type of t'R, while p, v need not do so; u, v cannot, however, be null in all types, by *300'17. *30302. 0q = '0/"NC induct Df *303-03. oo q = s'/O"NC induct Df

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266 QUANTITY [PART VI *303-04. Rat = X{(v)q.t,Ove NC ind -v +0.X= vJ]If *303-05. Rat def= X {(aj,v~).It,vED'UA cPU. X= (,uv)~ til1' Df *303-1. I B (,a/v) S.-. (Hp' oj. (p, o-) Prm (p,v).f! R- A SP [(*303-01)] *303-11. F E~n) /=A[*303-1. *302-36] *303-13. F. ~ = Cnv'(v/p) [*303-1. *302-11] *303-14. F.o0/0= A [*303-1.*302-36] *303-15. F: R(0/v) S.2 PeNC ind - t0.f! R AIr CIS..vENC ifd -t'0. 2[! C'S x'(xRx) Denm. F. *302 14-38. *303 1. F B (0Ov) S v E NC ind - t'0. ft! R' A SO. [*301'2].vE6NC ind -t" ft0. RjA I GCS: DF. Prop *303-151. F:.R,SeRelnumid.:): R(0/v)S.=-. v e NC ind - ff0. BRe RI'IJ. S e Rel irnm id. a! C'BR CIS [*303,15. *300-324-3] *303-16. F:R(,t/0)S.=E.,ttENCind-tf0.A!jSA1rC'R. -,NC ind - ff10. a! C']? n ' (xSx) [*303-15-13] *303-161. F:.B,SERelnumid.D:RGL4O0)S.=-. PE NC ind - t"0.1 Re Rel num id. S e RI'I. a! C']? A C'S [*303-151-13] *303-17. F:.p~, veNCind-tf0.R,SERelnumid.R(/ujv)S.:): R, SeRilll.v. R, SERel num Dem. F. *303-1. *113-602.)D F:: Hp.)D:. R, S eRel num id: (Hp, a-). p, a- e NC ind - ff0. ft! RO A SP':. [*300-33.*301'3]):. SEc Rel nuin id:. Re RI'I: (2[p). p E NC ind - ff0. ft! B A Si': v: BEc Rel num:(Hp, a-).,a- E NC ind - tf0. ft! Br A SP':. [*0003]):. Se Rel num id:. R e R1'I[. ft! IASI,. v. B E Rel num. ft! JAS,0:. [*300-333] D:.R, SEcRl'II. v.BR,SecRel num::) F. Prop

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SECTION A] RAITIOS 267 *c303-18. F-:.,, veD"UUtt,3CR.R, SeRRl'I. R (SClv)IO. N. R (O/P S. R (267) S. =.. a CIR r OS [*303K-151516. *301-3] *303-181. F! (Ft/ v). p (up, a). (p, a) Prmn (Ft, v) Dern. F. *3013.*300-325 17.) F (p, a) Prm (,t, v).). (ax). (xx, x) (,a/v) (x, x) (2) F. (1). (2).:) F. Prop In the above proposition, if /ulv is typically indefinite, so that " p! Ft/i"' oniy asserts existence in a sufficiently high type, p, a may also be typically indefinite. But if Ft/v is to be taken in a definite type, p and a must be taken in the corresponding type, and must not be null in that type. This is proved later. *3034182. F:. 0/0 = F/V.: (u, v e NCind). v. t = v = 0 Here the equation 0/0 = Ft/v is assumed to hold in a sufficiently high type. Dewtz. F. *303-14.) F:. 0/0 = F/v.) Ft/v = AO: [*303-181. *302836]: r (F, e E NC ind -ffO). v. Ft= v = 0 (1) F.(1). *303-11-14. F. Prop *303-19. F: R (F/v) S..B(Ft/v)S [*303-1. *121-26] *3032. F:.(p, a) Prm (/u, v).: R (1/v) S -.= R[2ASP Dem. *.303-1 D F~ Hp. Rr A SP. R (b~lV) S (1) F.*302-38. *303-1.

F: Hp. R (Ft/v)S.).~! Ro ASP (2) F. (1). (2). DF. Prop *303-21. F:. p Prm a.): R(p/ a)S = J& A.f! SP [*30231. *303-1] *303-211. F: (p, a) Prm (is, v). D. t/v = p/a [*303-2-21] *303-22. F: p Prm a. 0, v e NC ind. (, = v= 0). /i xca= v x,, p..t/V = p/a [*30237. *303-211] *303-23. F: /U, v, %, qe NC ind (p v = 0). i= = 0).,u x,, q v x Ft/v = 5/ [*3023. *303-211] *303-24. F:Fu, ve NC ind.r ((= v=0). D. (alp, a). p Prm.ao.Ft/v = pla [*303-211. *302-22] rhe following propositions give typically definite existence-theorems for ratios.

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268 QUANTITY [PART VI *303-25. F:.pPrmoa.): t!(plo-) t'R.u-.p, o-EC'(U~t3'R) =. p(R), o-(R)eC'U I.e. if p Prin a-, there are relations of the same type as 1 and having the ratio p/a- when, and only when, the number of relations of the same type as R is at least as great as p and at least as great as a-. Dem. F.*303-21.) F:. Hp.:): f! (p/a-)C t'B. D. (EIS, T). E! So'. E TP.S, Te t'R. [*301-16] D.p, a-E C'U t3'R (1) F.*301-16-3. F:. Hp.): p,a e C'UCt3B. X e t0'C'". D. (x 4x)p = (x x)o, = x, x (2) F.(2).*303-21.D F:. Hp. D: p, o- e C" UC OR. x,6 t,,CC'R. ~.(x X I) (P/O) (x I X) (3) F.(1).(3).*63-18. DF. Prop *c303-251.~:,veC CPR.N(~ =v=0 ~I> ~ Dem. F. *302-36-39. D F: Hp. D. (gjp, a). (p, a-) Prm (/i, v). AL, p. v a-. [*117-32] D. (up, a-). (p, a-) Prm (pt, v). p, a E C'U t3'1?. [*303-211-25] D. f(Ft!/v)t t'R: D F. Prop *303-252. F:PeNC indrCIUUt "C'R..-,(I. =v==0....!(i zlv))t'R Dem. F.*64-51-55.) F: ==Nc'a. ca et"R.xet,0"R.). 4,xCaaEAt t2'R (1) F.(I).*30014.)F:llp.D. v eCUCUt3'R (2) F.(2).*303 251. D F. Prop In the above proof, pL, v are assumed to be typically indefinite. If they are typically definite, sm"At and sm"v must be substituted for,a and v on the right-hand side of (1) and (2). The hypothesis ",a, v e NC ind A GUC t2LC'R" is a convenient abbreviation for ((,au, Pe~ NC ind. iz n t'U'R, v n t'C'R e OU UCt2CC C R By *65-13, 1. r tt'G'Re C1 U t2'G'R. t C t'C'R. E C'UC t2lC R. AtE C' U t2ZG'R. But "At E C' UC t2'O'R J requires that At should be typically definite, whereas Et NC ind" requires that At should be typically indefinite. Hence the hypothesis of *303-252 is only defensible as an abbreviation, meaning "Fzv eNCind, and if A, v are given the suitable typical definition, they become members of C'U t2'C'R." *303-253. F:At,,e NCindAG'U t2'X. (At=v==0).).!(At/v)! t0/ X [*303-252] *303-254. F: t, v eNCind.AA,VkEC'U. -(A=V0)=.). (At/V) too'X [*303253. (*65-01)]

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SECTION A] RATIOS 269 *303-26. F~e~n~=))(X.!~v~0' [*303-254. *300-17] *303-27. F: t v eNC in. p,uxvA EU'U.). pv=O/./vA [*302-15.*303-1] *303-3. F:p Prm a..PPxcT!Ir. D.PP (p/a-) P~ Dem. [*303-21] D.PP (P/0-) P~ (2) F.*301-2.:)F: Hp.p=O.)P=r"=~x,'f~rc (3) F.*302-14.)F:Hp.p=O0.).a-'=1. [*301-2]:). P17=P (4) F. (3). (4). D F: Up. p 0.:). ft! (PP),r A (PcT)P. [*303-21]). pp (/- PT (5) Similarly F:Hp.-0.).PP p-PT (6) F.(2). (5). (6).)I. Prop *303-31. F:pPrmo-.p+0.a-+0.(pxca)At'XePIU.:). (aP). PCe Rel num nt00'X. Pp (p/a-) Pw Dern. F -*300-46. *301-4.) F:Hp.) -(HP).P C Rel num.(B,'P)P~Pxca (B'(P) (1) F. (1). *303-3. DF.

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Prop P, Q ctoo/X Q CP.B'(P =B'Q. B'P=B'fQ Dem. F.*117-22) F:Hp. Pe(p +o,1)r.). (aa). a C CP. aea-+o1 (2) F. *261-26. *205-732.)D F: Hp. P C (p +0, 1),. a C U'P. a e a-+1,8= (a - t'Minp'a - t'maxp'cC) wv tB'P wv fB'P.:). fi a- +1 [*250-141.*202-55])P /3C(a- +c1)'r (3) F.(1).(2). (3).*205 -55.) F.Prop *303-32. F:pPrma-.p>.a-. a-+0.pk cU'IU.) Dem. ft ~~~~~! (p/a-) ~ (Rel num n t00'X) A RS (RP0 C. Sp0) F. *303-311.)DF:Hp. D).(HP, Q).Pe (p +o1)r.QC6(a- +oI)rP, Q'Et.'X F *300-44-45. *301-4.)DQC ' " ' ' 1 F: Hp..P(p +o,1), S=P.:). SeRelnum. (B'P)SP (B'P)(2 (2)

 Similarly F:Hp.QE (a- i -XR = Ql.)RE Relnu n. (BI Q) RcT (BIQ) (3) F.(I). (2). (3). *261-35-212. F: Hp. D. (aR, S).R, Se Rel num nt.'X. R1. C S,0. ft! KO A~S (4) F. (4). *303-21.)D F. Prop *303-321. F: pPrm a. ptO+ -t. +0-pA,LTA eU'U.).!!(p/u) ~(Rel num ntoo'X) [*30.3-32-13] *303-322. F:: pPrmna-. pA, akACDI'U P fIU.) D.~ (pa(Rel n tin ritoo'X) [*303-321] *303-323. F: p~, v e NC ind - tff0. D (ax).!(pulv) ~ (Ret num rn t.4/X) [*303-322] *303-324. F:, ve NC ind. k, vkE D"U.c ([tPrm v).D. Dem. F. *302-22. DF: Hp. D. (ap, a-, -i). p Prm a-. p j=0. a-t O.T +tO.r41.,u=p, T~. v=a- X0'. a! LA~ - a! P~A [*303-2-21] D. (up, a-). p Prm a-. p tO. a-tO + 0 -)V= p/a- a! (p ~0, I)A.!(a- +" I)A. [*303-321]) D. f! (,u/v) ~ Rel num: D F. Prop In order to the existence of (it/v) ~ Rel num in any given type, it is by no means necessary to have 1.k, v E D' U in the corresponding type. If p Prm a- - p, a- D' U n U' U, (p x. r)/(a- x. T) will exist, however great -rmay be, because (p X0, T)/(a- X0~ T) =p/a-. *303-33. F:~!(Lv~ (e unt (gp, a-). (p, a-) Prm (Ft, v). pA, a-AkE D' U ri'G U Dem. F. *303-322-211) F:(p,ao-) Prm (p,v).pA, akED'UAUI'U.).ft!(,t/v)~4Relnuimnt.'X) (1) F.- *303-181-15-16-211.)3 F:. ft! (ps/v)~ (Rel numrn At00'X).)D: (up, a-). (p, a-) Prm (p, v). p to0.+ a-t. ft! (p/a-) ~ (Rel numn A t.0X): [*303-21]) D (sip, a-). (p, a-) Prm (,a, v). p t 0. a- [0: (aR, S). R, S E Rel nUM A, t()-X. -R! A SP: [*301-41]): (ap, a-). (p, a-) Prm (p, v). p # 0. a-=+ 0. a!(p-'01)At'X~a!a-~01Ato'X (2) F. (1). (2). DF. Prop *303-331. F:.pPrmna-.:):f!(p/a-)~ (RetInum At00'X).=.pk,a-Ae D" ' UA [*303-33 - *302-31]

 *303-332. F':.-pPrmoa.):! (p/oj ~(Rel num nt,11'p).p,ae D U nG' U [*303W331] In this proposition, p, a- are typically definite cardinals, whereas in *303-331 they are typically indefinite. *303-34. I-: p,a-e6NCind. pA,0-A eDI'UA PI'U. I> a-.) (aR, S). R, S eRelnDUM nt.'X. f! RQASP.4aIf! R?1A MI Note that {-, I! Rn A ~StJ does riot imply E! -Rn or E! St. Dem. F. *303-311.:) F: Hp.) D.(HP, Q, R,S). P e(p+1),. Q e(a- +o1)r, P,Qet,,'X.B',P=B',Q.B'P==B'Q.R=P1. S=QI (1) As in *303-32 Dem, F. (1).)D F: Hp. D. (HP, Q, R, 5). Pe(p +0 1),. QE (a- +0, 1),. S= P1. R =Q1. R, Se Rel num. (B'P) (R,7 ASP) (B'P). [*121-48.*202-181.*301-4.*300-44]:).(a-R, 8). -R, Se Rel num. t,"!R ' A SP* (. R'1): Prop *303-341. F: PA, aAe eDUrdCl'U. pPrm a-. ~ Prmq. (p/a-) t.,/X=(/n)~ t0/IX.). Dern. F. *303-34-21.)F: pA- a-A E D"'UA (P1U. p Prm a-. Prmqi. q >o (p/a-) ~ tOo'"X +

http://quod.lib.umich.edu/cgi/t/text/text-idx?c...stmath;rgn=main;view=text;idno=AAT3201.0003.001 (138 of 246) [5/26/2008 7:31:51 PM]

(UI'q) W>t0' (1) F(1). Transp. *302-1.)DF:Hp.)D.iq a- (2) F. (2). *303-13. D)F: Hp.)D.4~~p (3) F. (2). (:3). *117-32.)F:Hp.).4:A,a-AeG(1'U (4) F. *303-322.)F: Hp.). f! (4/n) ~Relnum. [*303-11-15-16] D. ~ + 0.q + #0 (5) F. (2). (4). (5). F F: Hp). ~,k, nAEDU Un PIU. F. (2).(3).(6).:)F. Prop *303-35. F AEU Pm Dem. F. *300 14.) F: Hp.:).(ax, y). X y X, yE tD'X. [*303-16-17.Transp]) = 0. (1) F. (1). (2).) F.Prop

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272 QUANTITY [PART VI *303-36. F:.pA,o-,kc[GU -v.4:A,flke(CI'U: pPrmao-. 4Prm '7::): (p/a-) to' = (4:n) t00'X.. p 4 a q Dem. F-.*3 00 14. *3 02 K14.) F p pA, o-Ac (I (U. p Prm a. '~(pk, a-,E D'U).U) p= 0. a-= 1. vp = 1a- 0: [*30 3-3 513]): 4:Prmq. (p/as) t~ooX t4:/n) tX. p= 4:. a= (1) F.-(1).-*303'341.)I. Prop *303-37. F:.a,,ENCirindAP(U~t2I'X),.rd<a=13=0).v. (3E NC ind ^(IP(U ~ t2'X). _ (y= a3= 0)::): (a/l8) ~ t00'X-:(,y/() ~t00'X.)D. a x. (= 13 x. 1' Dem. F. *302-36. *303-211.)DF: a,13 isNC ind. a,flAEU ' U. c (a -f==0).)D. (Hp, a-). (p, a-) Prrn (a,/3). p/a- = a/fl (1) F. (1). *303-254K181. D F: Hp (1). (a/l8) ~too CA,= Qy/8) too'X.).D F.(1). (2). *302-21P22. *303-211.) F Hp (2).). (up, a-4, n,,) (p, a-) Prm (a, 13). (4:, n) Prm (ry, 3). p, a-e (ICi U. p/a- = a//3 = y/(3 = 4:/n [*303-36] D. (ulp, a-). (p, a-) Prm (a,13). (p, a-) Prm (y, (3). [*302-34]:). a x0(3=/3 xy (3) Similarly F: sy, (3E NC ind. 'yk,8, (3 I eG U.,-.., (vy = (= 0). (a/13) too'X = (ry/S) t00'X.)D. ax0(3=flx.r (4) F. (3). (4). D F. Prop *303-371. F:a,Aly,(3eNCind.ak,13A,, yk, 8x eG'U.,-.,(a Prml. y Prm 8). [Proof as in *3003-37] *303-38. F:.a,13,'y, (eNCind:aA,fl,,EGI' U. v y,\ A,&EU'U: rl- (a =13 =0). e- (y =(=0):) (a/f) ~ (ry(3)~ t~/X. a 0 (33x y[*303-37-23] *303-381. F:. a'fl'y,3e NC ind.ax,/3A,yk,(3A cC'U. r (a P1-Mfl. y Pr-m).)D: (a/fl) tOO'X =(7y/() tOOC X * a x(=fl x,,y [*303-371P23] *303-39. F:.a,1,8,y,(eNCind.-(a=fl=0).-(,y=(3=0).): a/fl=ry/(3.=.ax0(38=flx~ry [*303-38.*300'18] *303-391. F.~lNida~le'~~a1=)) [*303-38-254-11-14]

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SECTION A] RATIOS 273 Thus when a/fl is used as a typically indefinite symbol, we obtain the same results as if we supposed it defined as of a type t.fX, where a +0 1 and 3 +,,1 both exist in the type of X, i.e. Nct,'X > a. Nc't0',X >13. *303-392. F:.a,/3eU'U.-N(a=13=0).):(a/1)~ t1'c =(y/6)11'a al// = ey/8..a x03 = 8 xoy [*30339127] This proposition differs from *303391 by the fact that a, 3 have become typically definite. It will be observed that even when a and 1 are typically definite, al, like a x,,/, remains typically indefinite. *303-4. F:. p Prm a. R e Rel num.):1R,(p/o-) R,.=-. f R,,,, [*303'3-21. *301-4] *303-41. F:: ~u, ve NC ind. c (, = 0. = 0).):. R eRel num,. 4= 1cm (,, v).) R,(jp/v) IL. ~! Rf Dem. F. *303-2. *300-44. F:. Hp. pj= 0. v # 0. Re Rel num. (p, a-) Prm, (p, v).): [*302-37] Ria,,,a (1) F.(2). *302-22.)D F:. Hp. p #0. v + 0. R e Rel num. 1=cm (,a, v). D: R, (fklv) R., =! Rt (3) F. *302-44.) F:. Hp. /, = 0. Re Rel num.4 Icm (a, v).: O: [*303-15] D, tl)R e (4) Similarly F:. Hp. v=0. RE Rel num. =lcm, (, v).): R, (L/v)Rv. R, RI (5) F.(3). (4). (5). D)F.Prop *303-42. F:. Hp *303-41.:= 1cm (p, v).: U, (Hav) U1,.

Icm (,tt, v) 6 CA U [*303-41. *300-26] *303-43. F::. Infin ax.):,a, v E NC ind.cN (a = v = 0).),, U,(Hlv) U, [*303-42. *300-14] *303-44. F:. Hp *303A42. PE Ser.: P, (,uv)P =P,. t Pt [*303-41. *300-44] *303-45. F: Pe f infin.p, v e NC ind. 0 (.v = 0v 0).). P,& (av) P, [*300-44. *303-44] *303-46. F:. (p, a-) Prm (p, v). 4,fl E NC ind. Re Rel num. R(ct/V) Rq. 4: X0 a- = 7) Xe p. -R! Dem. F. *303-211.) F:. Hp.:) R, (pltv) R,.. R, (pla) R,. [*303-21] ft! Rtxx17 A RqXcP. [*300'55] x.:xa -=' qx p. ft! Rjxr0:.) F. Prop R. &w. III. 18

274 QUANTITY [PART VI *303-461.:v,,fNid.(= 0)e (==0Reeum: Dem. F. *302-45.) -.*302-35.) F:Hp.(p, a-) Prmn(at, v). 4x.a- =fl7x.p..(p,ao-) Prm (4:,). (2) [*302-34] X.4:xvj0,q. - (3) F.*302-35-37.) F (1).(2). (3). (4).*30.342. DF. Prop *303-47. F:.Hp*303-461.A,,EPot'(R.): R~(,tt/v)R,2.=. x,,v=, x0, [*303-461] *303-471. F.,~Nidr.(~=)r~4==)Pfifn) Pe,a/) P 4:, xvy x0 [*303-47. *300-44] [*303-461. *300-26] Dem. F. *303-15.)F:. z,v, 4, ENCind. = 0. v0.:): [*1 20-42] E:0. [*113-602] E.4xO7=flx0 (1) Similarly F:.p, v,:,,qcNC indq + 0. v= 0.: Ut (t/v) U,. x.4xv =f x0 (2) F. (1). (2). *303-48.) F. Prop *303-5. F: p,uaeNC ind -t'0.a! (p +Oo)A (ap, Q). P e (p +0 1),. Q e (a- +0 1),. P, Q EtoX B'P= B'Q. B'P=B'Q. G 'Pri ('Q-t'B'P u tcB'P Dem. F. *110-202. *120-417.) F:Hp.).(Ra,i3). a,I8 ctA'. ac p~ +01. e-8 o-0I.czarn8=A (1)

SECTION A] RATIOS 275 F *262-2.) F- Hp. a, /3 E t0,'X. a e p +I.3 — I. a n /3 = A. ar #2.)D. (Hp, 5). P, S el n t00'x. (7'P = a. C'S = 3. a no/ = A. [*251K131P141]). (HP, S,Q).P, S, QE f n to'X. C'P~a CIS = la Q =B'P+1-S -f +B'P. C'P nOQ=L'B'P v tB'Q (2) F. *262-2.) F: Hp. a,/,E t0'"X. acp +01I.~3 = tffx. xr' a 2 cr 2.) (H{P)Q).P, Q cto0'fX. PefLGUP =a. Q =(B'P) I x -t*B'P (3) F-.(1). (2). (3).) F. Prop *303-51. F:.pPrmcr.ptO.ot0O.H!(p~,o-)A~.): (MR, 5): R, SEc Rel nu m A t00"X.1? (p/r) S: ~1/ qt p/cr.)-R (Uq) S Dent. F-. *300-44-45. *301-4.)D F-: Hp.- P c (p +0 1),. Q E (a +0 1)r.- S= P1. 1? = Qj. B'P= B'Q. B'P= BQ. GCP ri C'Q = VB'Pv t'B'P.).~!R~atSP (1) F. *301P41.)F: Hp (1) ~p.nq).). R17AS = (2) F. (1). (2) *303-21.) F:. Hip (1).): RB(p/c) 8: Prmq. (p p.q = a-)) 1 (~/,q) 5: [*302-22.*303-211]): 1? (p/cr) 8: 4,,q c C' U. (= = = 0). ~, t p/cr)-.I 1? (4Jq) 5: [*303-182])D: R (p/cr) 5: ~1/y + p/cr. D),. B- (~/y) S (3) F. (3). *300-44. *303-5.) F. Prop *303-52.Fq&,Nid-'.(z~v.) (SR, 8): R, S E t0'cX. R (lk4v) 5: ~/, t i/v. DI,, B (~/q) S Dem. F. *303-24. *302-39.)D F: Hp.D. (H{p, -). pPrm ar.)/ V = p/Or. p 0. + 0.a!p~c (1 F. (1).*303-51.)DF. Prop *303-6. F: v E NC ind -t'0.). 0/v= Oq [*303-1-5] *303-61. F: veNCind -t'0.D).v/0=ooq [*303-16] *303-62. F.0q = Cnv'looq=R8(['BAIrG'S) [*303-6 61-1315] *303-621. F. Or Rel num id = Cnv'(Rel num id 1 00 q) - IS (R I. S e Rel num id.! C'B AC'S) [*303-6-61,13-151] 18-2

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276 QUANTITY [PART VI *303-63. 1-:H!2x.).fj!Oqr(Relnuin n t0'x) Dem. F-. *303-156.D)F x ~y.D).1IOq (XI 4 y):) I-. Prop *303-631. F a! 2x-..t! (Rel numnAt0I~0'x 10q [*303-63i62] *303-65. F:H2A.:).Oq~t/XooqX+ 2tOO'~x Dem. -. *.303-62.) I: XtY.). Iq(X 4,Y). {I 10q(X 4, y)}) IF. Prop *303-66. F:2!AD(/) t'=q=.~=.e~n-t Dem. F.*303-6.) F: /=0. vENC ind - t'O.)/v = Oq (1) F-. *303'6 15.-) F-:pltv=Oqn:).tLI/v=RS (R e RLI. SE Rel nu mid. a! C'RA G'CS) (2) F-.*300-3. F::Hp.:). (Hx,y).x~y.x4,yeRelnum nt.'X. [*10-24]:). aj! (Rel num id - RII'I) n t~'X (3) F. (2). (3). *303-11-17.) F.(2). (3).*303 16.) F:. Hp. D: (pv)~ t.6X=0q. D.r,(ittO. P=0) (5) F.(4). (55). F:.Hp.D: (,u/v) ~t00"X = 09.D.jt0. v eNC ind - t' (6) F.(I).(6). F.Prop *303'67. F,):(/)~t" =o v0-peN n " [*303-66-62] *303-7. F: XE Rat ~(g, v).,a, v eNC ind v#+0.X = p/ [(*303-04)] *303-71. F: X eRat def.ij, v).,, v eD' U GIU. X =(p/)tl,' [(*303-05)] *303-72. FXEa..(t)!Xt1L [*303-26] *303-721. F:Xc Rat - t0l.). X ~t11'1tteRat def [*300-18. *303-771] *30373. F:Xe Rat def.).! X ~Rel num [*303-322-324] *3031731. F:.-pPrm o —:): (p/o-) ~t,'pe Rat def.~ p, o-eDI'U n(WU [*303'71. *302-39]

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SECTION A] RATIOS 277 *303-74. F:. pPrm a. X=(p/oa-) tl'p.):!X Relnum.-.p, aoeD'Un ('U [*303-332] *303'75. F! X C (tll' n Rel num).. X C tl,' e Rat def [*303-74-71] *303-76. F:. X, Ye Rat.X t,1'p e Ratdef.:: X t1'p = Y t1'p.. X= Y [*303-391] *303'77. F:. Infin ax.:):., v e NC ind - 'O. D. u/v e Rat def [*300-14. *303'71] *303-78. F: Infin ax.). Rat def= Rat - 'O [*303-7-77] The above two propositions assume that /Lv in the first, and "Rat" in the second, have been made typically definite, but they hold however the type may be defined.

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*304. THE SERIES OF RATIOS. Summary of *304. In this number we consider the relation of greater and less among ratios, and the series generated by this relation. We need two different notations, one for greater and less between typically indefinite ratios, the other for greater and less between ratios of the same type. The former is more useful where we are dealing merely with inequalities between specified ratios, but the latter is necessary when we wish to consider the series of ratios in order of magnitude, since a series must be composed of terms which are all of the same type. We put *304'01. X<r - =.(, v, p, -)., v, p,e a NCind.o-0.p xc < v xcp. X=-/z. Y= p/ar Df This definition is so framed as to include Oq but exclude oo q. For the relation "less than" among rationals of given type (excluding Oq), we use the letter H, to suggest r (defined in *273), because, if the axiom of infinity holds, the series of rationals of a given type is an a. The definition is *304-02. H = XY {X, Y e Rat def. X <, Y} Df When we wish to include 0q in the series, we use the notation H'; thus *304'03. H'= XY

{X, Ye Rat def v t'Oq. X <r Y} Df (It will be observed that here t'Oq acquires typical definiteness through the fact that it must be of the same type as "Ratdef" in order to make "Rat def v 'Oq " significant.) If the axiom of infinity does not hold, H and H' will be finite series: if v +c 1 is the greatest integer in a given type (v > 1), the first term of H is 1/v and the last is v/I (*304-281). In a higher type, we shall get a larger series for H, but at no stage shall we get an infinite series. If, on the other hand, the axiom of infinity does hold, H is a compact series (*304'3) without beginning or end (*304'31) and having No terms in its field (*304'32), i.e. H is an 7 (*304-33). In this case, C'Hf=D'H=Rat - 'O (*304'34), i.e. any rational other than 0q, as soon as it is made typically definite, belongs to C'H.

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 Under all circumstances, HI is a series (*304-23), and H exists in the type t00,'X if 3 exists in the type t'X (*304'27). In the same case, G'H =Rat def (*304-28). Similar propositions hold for H'. C']]' consists of typically definite ratios, and if X is any ratio, there are types in which X belongs to C'HR' (*304-52). If the axiom of infinity holds, every ratio is a member of G'H' in every type (*304-49). *304-01. X<,Y.=.(Hfj v,p,a")./ h4,v,p,a"ENCind.o-tO./Lxca"o<v x op. X= 14v.Y = p/a Df *304-02. H = X tX, YE Rat def. X <. Y} Df *304-03. Hf='Y'I{X, YERatdefvt'0q.X<rY} Df *304-1. F: X <rY. E.(Of/,v, p,a").-,ctI) P) "NC ind. pxca"o-< vxo,,p. X=-,a/v. Y= p/a" [(*304-01)] *304"12. F X <r.y. E Y<rX [*304-11. *303-13] *304"13. F: X<r Y-:).-XYeRat. Y +0q Dern. F. *117-5. F:4 x0, a" < v x,,p.. v X,, p to. [*113-602] V)#. p+0 (F. (1). *304'1. *303-7 D F. Prop *304-14. F:XJIY.=. XYE Rat def. X<r y [(*304"02)] *304"15. F:H..~~~~~".,~~~DUd1U X = (a/IV) ~ t,16',. Y= (P/a") ~ t11',t. ~L xC0 a < v x0.p [*304-14'1. *303"71] *304"151. F: XFIY. (HMN~p). M <rAT. M~ tilj,u,N~tli'/.eRat def. *304"152. F:.,uAPrm v. pPrma".:): f(,/v) ~tli1' H {(p/a") tilpl j1V <r P/a". u, V,p(,oa" ED'UnA(I'U [*304-151.*303-731] *304-16. F,/)Hpa) a/)Hv~) [*30415a] *304-161. F: XHY. E.YHX [*304-12-151] *304"2. F.H CJ 1) Dem. F.-*303-37.-) F: p, v, p, a" e D'" U A (III U. (lk4v) ~ t, =. (p/a") ~ tili..). U X0 a" V X0 p. [*304"15]) (pt/v) H (p/a")} (1) F. (1). Transp.:) F. Prop *304-201. F. (X <rX) [Proof as in *304'2]

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 *304-21. F. e E trans Dem. F.*304-15. F: XHY. YHZ.). (gizL, vP,p,, a-, v ~, p), o-, ~,, e DI U n GI U., x,, o < v Xe P F. *117571. *120-51.) F: P, v p, a-, 7q e D' U A (, U. pa xe a- < v x, p - p xq, < a0 x-. x Ft X0 a X0 fl <v7 X0 p X0 fl < V X0 a7 X0 k [*126-51] D. t xn, q < v x,04 (2) F.(1). (2).)DF. Prop *304-211. F: X <, Y. Y <, Z. D. X <, Z [Proof as in *304-21] *304-22. F. H e connex Dem. F. *12633. DF:. F i,p, -a E D'Un A'U. D: LU X0a- <v X p.v.,UXoa- =vX p.v. tXeU>vxcp (1) F.(1). *304-15. D F.Prop *304-221. F:.X, YERat.):X <, Y. v. X= Y. v. Y<,X [Proof as in *:304-22] *304-23. F.HeSer [*304-2-21L22] *304-24. F:,wz'eD'UAG'U.v [.).(Ftv)H{Ft/(V- 1)J Dem. F. *120414415416.) F Hp.). v-,I e D'U [U (1) F.(1). *304-15.)D F Prop *304241. F:FteD'U.,L~0leG'U.).(Ft/1)IHt(Ft +01)/11 Dem. *.~300-14: Hp..,aC, le CLI UI F.*300-14. *120-124.)F:Hp.) i~1E D'U (2) F.(1).(2). *304-15. D F. Prop *304-25. F:WVeD' U'U.C (Ft+Cl =B'U.v=1).). ~tlveD'H.v/1e eU'H [*304-24-241-16] *304-251. F:Ft+,1=B'U.).pu/1l,eD'H -Dem. F.*300-14.) F: Hp.p, a eD' U n GIU.)D.p ~. 1 a. [~c111~.?71] ~ ~ P Xo 1 ~ tL Xe a(1) F.(1). *304-15. DF. Prop *30426. F:pPrm v.): 1/v e D'H..V/vleU c'H. [323, *3e D0 U n 42U.,-2V 511 +C 16]U. v = 1) [*A302-39. *1304-~25-251-15-16]

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SECTION A] SECTION A] THE SERIES OF RATIOS28 281 *304-261. FDHX(~~), ~e'rUU'j~lBUv) X = (pjv) ti tplI. [*304-25-25 115] *304-262. F. G H= A{(yv) qA1, ve D'U U G'U. r-,(jt+01 = B'U. v= 1). X = (vbA)~ til1'14 [*304-261-16] *304-27. F:ft!H.=-a!3 Dem. F *300-14.:) [*304-25] 1.i.).av (1)n(I.(, BU F *304-261.)D F:. f [! H. (Hu, v): /, v E D'Ur P U: v 0 eP. v + I $1 [*117-32])a! 3 (2) F.(1).-(2).) FProp *304-28. F:!..'={~.~~tvDUUUX~/)t'. =Rat def Dem. F.(I)0-1.)DF:.Hpp. (~tv. D: +C1=BI'UBt. D. v+1=BU.>=1 (2 F. (2). *304-261-262. *303-71.)F. Prop [*304-28261-262] *304-282. F. E G'H60 [*304-27-28.*303-66] *304-29. F(4)Hpa) u~0~+oeU) Dem. ~~~(,4lv) H {(p +. p)/(v +D oajj. ftki + p)/(v +oa-)} H (p/a-) F.*304-1. F: Hp.). ptxa-<v xp. [*126-5] X./t (v~+0,a-) < v X,,0(A+0p). (a +0 p) X. < (v +0,a-) Xp. (1 F, (1).*304-1.)3F. Prop *304-3. F: Infin ax.). H e Ser rn comp [*304k29-23] *304-31. F:Infin ax.).et-.'E! B'H. '~'E! B'H [*304-281. *300-14] *304-32. F: Infin ax. C) (.'H No Dem. F.*304-15. *303-211.*302-22) F. Nc'C'H < Nc' ^ {(Hp, a-).- p Prm a-. p, a- e D,'U ^(1 PU. X=p/o-} [*.30.336] < NclfM{(3Ip) -).p Prm o- p, a- eD' U nPU. M p=4a-J [*33,161] <~ Nc'C'U x0Nc'CG'U(1

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282 QUANTITY [PART VI F. (1). *123-52. *300721.) I Hp.) NcP'C'H~< N, (2) I -. *304-28..) F: Hp. Nc'C'H,> Nc'X {(Hv). v E D' U n (4ZUU. X = v/1l} [*303-36] >, Nc'(D'I U n P(U) [*300-21]?,No(3) F.(2).(3). *11 723. DF. Prop *304-33. F Infin ax.).H Eq2 [*304-3-31-32. *273-1] *304-34. FI- Ifin ax.).CGH =D'H =Rat -t'Oq [*303-78.*304-28] *304-4. F: XH'Y. -. X, YE Rat defy L'tOq. X<r Y. *304-401. F:. Infin ax.) X<r Y.-. XH'Y [*.304-4.*303-78] *304-41.FDH' {(,)FvUUv1:.(F+1BU=1) [Proof as in *304-261] *304-42. F.U''X{aLV).,, vE E1'U.FU+t0. V4O. X=(P-tV)~ tll'Fl *304-43. F:A!H'.=-.2!2 [*304-42] *304-44. F a2.)D. C'H'= X{(at, V). /L, vEU'lU. v#O. -X = (Fv) tul'1i [*304-41-42] *304-45. F: aL! 2.). B'H'= Oq [*304-4142. *303-6] *304'46. F: a! 3. D. H' = Oq +F H [*304-45-427-1] *304-47. F: Infin ax.:). H' e ii —~ [*304-46-33] *304-48. F. H'e Ser Dem. F. (1). *304-43-4623. D F. Prop *304-49. F:Infin ax.:). C'H'=-D'H' =Rat [*304-34-46] *304-5. F:X eC'H.D).ft! X ~Rel num [*303-73. *304-14] *304-51. F:XecC,'H'. D). q! X Rel num Dem. F. *30363.*304-43)F: Hp.).t! q rRe nuni (1 F. (1). *303-73. *304-4. D F. Prop *304-52. F: X c Rat.). (vtp). X ~ toiF E C'H' [*304-44. *300-18] *304-53. F:

X Rat -tL'Oq).(gjt).X ~ti'ltpECG'H [*304-28. *300-18]

*305. MULTIPLICATION OF SIMPLE RATIOS. Summary of *305. The ratios hitherto considered are called "simple" ratios in opposition to "generalized" ratios (introduced in *307), which include negative ratios. We deal with multiplication and addition first for simple ratios, and then for generalized ratios. In this number we are only concerned with the multiplication of simple ratios. In defining multiplication of ratios, we naturally frame our definition so as to secure that the product of u/v and p/a shall be (,L x p)/(v x,,). This is effected by the following definition (where 'Ls" stands for "simple"): *305 01. X x, Y= RS [(g/~, v, p,). i,v, p, p, e NC ind. v +O 0. a-: 0. X = /Iv. Y- p/la. R {(L XC p)/(v Xc a)} S] Df which gives us *305-142. F:,, p e NC ind. v + 0. - 0.:. /Iv X p/a = (, Xc p)/(v Xc a) and *305-144.: g! (,u/v x8 pla). /v x8 p/a- = (I Xo p)/(v xo a) The reason for the hypotheses in these propositions is that, if p is a cardinal which is not inductive, while p=0 and v, a- are inductive and not 0, P/v A and t/v x8 p/a- = A, but (/u xc p)/(v XC-) = OqFor the applications of the multiplication of ratios, it is essential that we should have, if' R, S, T belong to a suitable vector family, R (/v) S. S (p/ a) T. D. R (D/v x, p/a) T, e.g. we want two-thirds of five-sevenths of T to be (2/3 x8 5/7) of T. It will be shown in Section C that our definition satisfies this requirement. We prove in this number *305'3. F: X, Ye Rat.. X x8 Ye Rat *305-22. F:. X x,= Oq. =: X, YeRat: X-Oq v. Y= Oq i.e. a product only vanishes when one of its factors vanishes; *305'301. F: X, Ye Rat - t'Oq. -. X x, Ye Rat - t'Oq

284 QUANTITY [PART VI *305-25. F:p, v,p,a-,ED,'Ur'i PU.). (plv xsp/o-)~ to e C'H Thus a product of two ratios which both exist in a given type exists in the next type, i.e. *305-26. 1-: X, Y e Rat.- X ~ til'1c, Y ~ til'pe Rat def. D. (X x, Y) ~ t,,1peC'1H The formal laws offer no difficulty. We prove the commutative law (*305-11) and the associative law (*305A41); we prove that X x8 1/1 =X (*305-51) and that X x, X = 1/1 (*305-52). Division results from *305-61. IF:.AeRat-tf0q.A',eRat.:):A xX=A'.=E.X=A'xA and the axiom of Archimedes is given by *305-7. -: X, Ye Rat - tf'q)(3a). aeNC ind. Y<r(a/1 x, X) *305-01. XxY=RSI'[(,u,v,p,a-). L,UV,p,a-,ENCind.vtO.a-#0. X=,a/lv.Y=p/a-.R{(~txp)/(vx~ra-)}S] Df *305-1. F:R (XxY)S.=-.(gu,v,p,a-).p,v,p,a-ENCind.v+0.a-+0. X = ~ = p/a.]? {,c x#p)(v x0, a-)} S [(*305-01)] *305-11. F.Xx8,Y=Yx,,X [*305-1] *305-12. F: X, Y '"E t'0q V t' 00q* Cnv'(X x, YT) = X x, Y [*305-1. *303,13] (Ft X. p)/v X. a-) X.(p x~p)/(v' XC a-') Dem. F.*303-39.)F:Hp.). tx0 v' = vx0,tt'. px, a-' =p' x~,a-. [*120-51]) XCXO 1 xcv xa-'=I1tt' XOp xv XOa-. [*303-39] D. (Ft x0~ p)/(v X 0 a-) =(t/ X,, p')/(v' x0, a-'):) F. Prop *305-131. F:v),pa- NC ind - t'0.0/v =bt'/v'.P/a- = p'/a -'.) (0 X. p)/v X0- a-) = (k,~ X. p')/(v' Xc a-' Demn. F.*303-66.)F:Hp.).,'= 0.v',eNC ind -f (1) F.(1).*303-6. F: Hp.). (0 x,,p)/(vx,,c)= =0q([Lk' xp')/(v'xca')) F. Prop (t X~ p)/(xAV a.-) = (FX. xP,p)/(v' x0 a-') [*305-13-131]

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SECTION A] SECTION A] MULTIPLICATION OF SIMPLE RATIOS28 285 *305-14. F:,tt+0.p+0.v+0.o-+0.:)q.L/vx~p/a-=(,Lxp)/(v x.a-) Dem. F -.*305-1-132.) I-:: Hp.:):. R (t/v x,,p/a-)SE (Ht,U v', V/)p', a-'). p', v', p', a-' e NC ind.p~/v = U'/v'. p/ a- =p/-.v' #0+ - 0 +: R t(ip xO p)/(v x. a-)}S (1) F. *303,181. *302-36. *120-512.)D F:Hp. R (px,,p)/(v x,,a-)J S.)., tv,p,a-e NC ind (2) F. (1.(2). -) F. Prop The condition a~ +~ 0. p # 0 is required in the above proposition because if, e.g. 1-u 0. p E NC infin, we shall have (if v, a- NC mnd - t'0) PI/P =.q p/a- = A, whence,a/ v x~,. p/-=Alu ~p/vx - q f we assume ut, P e NC ind, it isnot necessary to assume ~ #0. p #0. This is stated in *3045142. *305-141. F:.v=0.va-c=0:).Dvxp/a-,/=A Dern. F.*303-6711.:)F:v=0.Ft',v',eNCind.pl/v=,tt'/v'.).v'=0 (1) F. (1).*305-1. F.. Prop *305-142. F:Ft,pENCind.v#0.a-t0.:).F/vx,,p/a-=(Itkx,,p)/(vxra-) [Proof as in *305-14] *305-143. F:ft!(t/vx.,p/a-).:).Dk,v,p,a-eNCind.vt0.a-#0 Dem. F. *305-1 F F: ft!(4/v x p/a-).:). (2,~v'). u', v'E6NC ind. V'#O. /v = p'/v'. [*303-182-67]:).p, v e N C ind. v + 0 (1) Similarly F: ft (Ft/n x,,p/a-). D. p,ao- eN Cind.a-#O0 (2) F. (1).(2).:)F.Prop *305-144. F: [! (Ft/v x8p/a-).).,ultv x8p/a- =(x,&tx0p)/(v x, a-) [*305-143-142] *305-15. F:.',(A,v,p,a-eNCind).v.,v=0.v.a-=0:).p/vx,,p/a-=A [*305-143. Transp] *305-16. F:.,v P) a-e NC ind: = 0. v. p= 0: v~0.ao-#+0:). Ft/ V X8 p/a- = Oq [*305-142. *303-6] *305-17. F. X x800q=A [*305-141.*303-67] *305-2. F:ft!Xx8Y.:).X,YeRat Dem. F. *305-1) F: Hp.:).:I v,p,a-).,, ~v,p,a-e6NCind. v40.a-+0.X=,u/v. Y=p/a-. [*303 7] D. X, Ye Rat:) F.Prop

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286 QUANTITY [PART VI *305-21. F:XxYERat-t'Oq.:).X,YERat-tfOq Dem. F.*303-72. *30.5-2.DF: Hp. D.X, Y eRat (1) F.*305-16. Transp.DF: Hp. D. X +09. Y{+Oq, (2) F-. (1). (2).) F-. Prop *305-22. F:.XxsY=0q.=:X,Ye6Rat:X=0q-v.Y=0q Dern. F-. *305-12-142. *303-66.)D I-:.X x8Y=Oq.: (3j1,v,7,p,a-).X==FIv. Y=p/a-.,u,v,p, a-eNCinld.,tk x0 p 0. v X, a- +0: [*303-66] ~ tvpa)X~v ~/-~vpae~rd iiO - 0. - + 0:,UIjV=O V. /O/-= =q [*30.37] *XYeRat:X=Oq-.v.Y=Oq:.DF.Prop *305-222. F: X x8 Ye Rat. D. X, YE Rat [*305-21-22] The following propositions are lemmas designed to show that if X, Y are ratios which exist in a given type, X x, Y exists in the next type. *305-23. F:pNCind.).(2 x0,+1 [*117-652.*120-429] *305-231. F /.(k +0l)2= p2+C (2 x at)~0,,1 [*1 16-34. *1 13-43-66] *305-232. F: e NC ind. p~2< 2P+-~c Dem. F. *116-311-321. D F 02 < 20+ol (1) F. *305-231. D Hp. /.2 < 2'~'A.I.).Q~ +1)2 < 2/1L~+c +a(2 x c /) +c, 1 (2) F. (2). *305-23.:)F:u Cid <2 +' + 2<24- 02+ F.(1).(3).JInduct.)D F.Prop *305-24. F:Ft,v,p,a-oeD'UnGUU.:). (Ot x. p) A t',u, (v x0 a-) nt e EDC'U (['1U Dem. [*305-232]:).,2 nt11jE (IU (1) F. *116-35.)F:Hp.:).p2ntcuED'U (2) Similarly F: Hp.). V2AritllFt'p2 nt~,tk,a-2Ant~,e DC U (['U (3) F.*117-571.:) F.H - (4) F. (1). (2). (3). (4).:) F. Prop

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SECTION A] SECTION A]MULTIPLICATION OF SIMPLE RATIOS27 287 *305-25. F la At,vp,o-e D'U n GI'U.)(Ft/v x,,plou) ~t.'AtEG'H Dern. F *305-14. F: Hp.).Atv xp/ a-= (p xp)/(v x,,o-) (1) F. (1).*304-28.-*305 24.) F. Prop *305-26. F: X, YE Rat. X ~ till'At, Y~ tl/ltzE Rat def. (X x8 Y) ~ too'At E C'H [*300525. *304-28] *305-27. F:X, YE Rat - t'cOq.) ([). (X x8. Y) ~ toO,'/4 E G'IH [*305-26. *303-721] *305-28. F:X, Ye Rat.). (:j). (X x8Y) ~t~oo'AtC GH' [*305-27-22] *305-3. F:X,YeRat.=-. XxYeRat Dem. F.*30'5142.*303'7.) F:X,Y~eRat.:).Xx8,YcRat (1) F. (1).*305-222.:) F. Prop *305-301. F: X, Ye Rat - t'Oq..,= X x, Ye Rat - t,'Oq [*305-142. *303-7. *305-21] *305-31. F ~L.X ~ tu'l ~, Y~ tll'At C C'H. (xrv).- (X x8, Y) t,1'v e C'H [*305-301. *304-53] *305-32. F: (HA). X ~ tu'Atk Y~ ti' C GH'..(a1v). (X x, Y) tll've 0 G'f [*305-3. *304502] *305-4. F:X, v, o-eNCind.t4 0. p +0. T 0.:). (X//tX 8 V/p) X i(0/T)=(X V X,, -)/(kX,ttxp X,,r) =X/At X,(V/P X,, -/T) [*305-142] *305-41. F.(X x8Y) x8Z =X x8(Y xZ) [*305A42] *305-5. F.).(b)~(/)=X~t[*305-14-142-15] *305-51. F: XcRat.).X x,(1/1)=X [*301515] *305-52. F:XeRat-t'Oqo).Xx8X=1/1 Dem. F.*305 14. *303 13.) F:Hp.). (lt,v). k, ve NCind -tf0. X x8X = (At X0 v)/ (v'X0F). [*303-23]) X x X = 1/1: F. Prop *305-6. F:.AERat-t'Oq.Xe6Rat.:):A xX=A'. =-.X=A'x8A Dewm. F. *304-1-4. *305-32-222.) F:Hp.).([att v, p, o-, ~,) AP, 0-e NCind - ff0. p, ~, e NCind. A=At/v.X=p~o-.A'=~1/n (1)

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 F. *305 142.) I-~:. ~,av,o-,eNCind - t'0 p, 4,ie eNCind.:): P/v x8 p/a- = 4:nt. (P X. p)/v X. O-) = 4:/n. [*303-38] -tk XcP Xc?=v XcO- Xc4 [*303-38]. p/0- = (v xc 4:)/(k x.q) [*305-142.*303-13] = 4:/n X~,, Cnv'(g4v) (2) F-.(1).(2).:)F. Prop *305-61. F:.AERat-t'0q.A'ERat.):AxX=A'.=. X=A'xsA [*305-6-222-32] *305-7. 1-:X, YeRat-tfOq.).(ga).aeNCind. Y<,.(ca/lxsX) Dem. F. *117-571. *120-511. *117-62.:) I- v, p, a-e~NCind - L'O. 4>v.:). 1tL Xe P'O X04 IT a> v' x0p. [*304-1]).(p/-) <r (Ft XX0 x 4)/v. [*305-14]:). (p/a-) <r {p/IV X, (p X~,,4)1 1 F. (1).*304-1. *120-5.)F.Prop *305171. F:. ZeRat - t'q.)X <r Y XXs Z <r Y XZ Dem. F. *305-142..) F:Hp. X <r Y) vH, p, a-, 4:, n'1).'u F) v, p, a-, 4, e NC ind. v + 0. a- + 0. 4:=O. O0 X = P/v. Y= P/aT- Z=4:/1n.FPxe O- < vXc P X XsZ = (PXc,)/(v X.1).Y X 8Z =(P X. O)/(aXcq). [*304-1.*126&51]). X X, Z <r Y X, Z (1) F.(1). DF:Hp. X X8Z<r YX8 Z X X, Z XZ<r Y XsZXs Z [*305-51-52] D X <r Y (2) F.(1).(2).) F.Prop

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*306. ADDITION OF SIMPLE RATIOS. Summary of *306. The addition of simple ratios is treated in a way analogous to that in which their multiplication is treated. We wish to secure that the sum of X/v and I/v shall be (X+c,)/v, and that the sum of 1/v and p/a shall be t(pa X a-) +, (v Xc p)J/(v Xc a). This is secured by the definition *306'01. X + Y= RS [(tv, v, p).,v, p e NC ind. v 0. X = p. Y= pv -. R {(L +c p)lv} S] Df whence we obtain *306 13. F: v $ 0. D.,/a + p/v = (+c p)/l *306-14. F: v 0. a-r 0. D. /v + p/- = {(S Xc a) +c (v Xc p)}/(v xc -) Our definition is so framed that ooq+gO q=A. This is on the whole convenient, though we could, of course, frame our definition so as to have 00 q +s 0 q = 00 q. In applications, if R, S, T are members of a suitable vector-family, we want to have R (plv) T. S (p/ la) T. D. (R I S) (4v, +, p/,a) T, e.g. if a vector R is 2/3 of T, and a vector S is 5/7 of T, we want the vector which consists of first travelling a distance R and then travelling a distance S to be (2/3 +8 5/7) of T. We shall show in Section C that our definition of addition fulfils this requirement. As in the case of products, the sum of two ratios is a ratio (*306"22), and the sum of two ratios which exist in a given type exists in the next type (*306'64). A ratio is unchanged by the addition of 0q (*306'24), and a sum of two ratios is only 0q if both the summands are 0q (*306'2). No difficulty is offered by the formal laws: we prove the commutative law (*306'11), the associative law (*306'31), and the distributive law (*306'41). An important proposition is *306-52. F:. X <r Y. =: X e Rat: (Z). Z e Rat - 'Oq. X + Z= Y When the axiom of infinity is assumed, this proposition becomes XH'Y.: X e C'HZ':(aZ).ZeC'H. X +, Z= Y. R. & W. III. 19

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290 QUANTITY [PART VI We prove also the proposition upon which subtraction depends, namely *306-54. F:. XYeRat.):X ~8gY= X~,Z. -. Y= Z *306-01. X+, Y=BRS[(H,v,p).,o,v,pcNCind.v40. X=f1/v.Y=p/v.-R{(,u+0p)/vJS] Df X=ptzv. p/v. R t(Ak ~,p)/v}S [(*306-01)] *306-11. F.X+,8Y= Y~8X [*306-1. *110-51] *306-12. I-: ft! (X +8 Y). D. X, YEc Rat [*306-1. *303-7] *306-121. F: tt/v = kt'/v'. p/v =p'/v'.).(Ot +0, p)/v (pk' +, p')/v' Dem. F. *303 39.) F:Hp.,ac, ii, p, ',Ik, p' e NC ind. v 4 0. V' + 0.),a X0 1) / X0 V. fJ X0 V' =p X0 v. [*113-43] D.(ik+0,p)xv = (a' +0p') X.v. [*303-39]:). (/k +0 p)/v (~ 0p')/v' F. *303-181. *302-36.) FHlp., tk v, p,,tt'', V) p' Cm) ~~ p)/v =.(W ~0 a/')-=A F.(1).(2). *303-67.) F. Prop *306-13. F: v4O.0).t/v +,p/v=(fL+p)/v Dem. F. *306-1)F: Hp.:). (t +p)/V C,ul/v +8 p/v F. *306-121. D F":/V =,i/v'.p/ v~p'/v'.X (~k'~0,,p')/V'}I Y.. X{Qt~p)/v} F.(2). *306-1.) F~ ~. p/v~8p/vC(+~0p)/v F. (1).(3).) F. Prop (1) (2) (1) (2) (3) *306-14. F: vtO.ot0.).Fu/v+,p/o'=t(Ftx~a-)~0(Vx,,p)}/(vx,,o-) Dern F *303-39. F:Hp.,u, vi, p, a-e NC ind..,ul = (Ft x. o-)/(v x0, a-). p/a- = (v x0, p)/(v xe a-). [*306-13] D)11 q/+,Rp/0-={F x a-) ~+v,)J(xa) 1 F.*306-12. *303-11.)DP (VX F.(1).(2).)DF.Prop *306-141. F:.v =0.v a= 0:).tv~48p/a-=A [*306 12.Transp.-*303 7]

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SECTION A] SECTION A] ADDITION OF SIMPLE RATIOS29 291 *306-15. F:,u/v+, p/a-=0q7.::-.,a=p=o.v,a-eNCind-t'0 Dern. F.*306-14.*303-66.)F:~t=p=0. v,a-eNCind-t'0.:).pj/v~p/a-=0q (I) F. *306-12.)F:/ /+8 p/a- = Oq.) tt, v,p, ce NC ind (2) F. *306-141.)F: u/v +S p/a-=Oq.).V +0. a-tO (3) F. (3). *306-14.)F: Hp (3).) {(, x0 a-) ~0 (v x0 p)}/(v xc a-) =0q [*303-66] Gk(~ x'Oa-) +-0(v xp) = 0. vX, x0a-=I0. [*110-62.*113-602] /). =p=0. V4.Oa-j=0 (4) F ()(2). (4).)F. Prop *306-16. -RXSYA[(H, v, p,T). u),v p,aTE NC ind.v +0. a- +0. X=14tv. Y=p/a-. IR t (, X0a-~ cV xe P)/V xc a-}S] [*306-14-12] *306-17. F:~t=0.v,p,a-e6NCind.vt0.a +t0.).,ulv~pla-=plaDem. F. *303-6.:) F: Hp. P /V =I' 0a. [*306-13].,a/V +8 p/ao-

University of Michigan Historical Math Collection: Principia mathematica, by Alfred North Whitehead ... and Bertrand Russell.

= (0 +c p)/a-:) F. Prop *306-2. F:X +-sY= 0q.. X =Oq. Y=0q [*306-15-12] *306-22. F:X~8YeRat.=-.X,YeRat Dem. F. *306-16. *303-7.:) F: X +8 Ye Rat. POT)p a-. /a, V, p, a-c NC nd. X = ~t4v. Y= p/a-. v x0, a-to.0 [*303-7] =. A, YeRat::)F. Prop *306-23. F:X+,sYeRat-t'0q.=.X, YeRat.,(X=Y=0q) [*306-22. *303-7. *306-2] *306-24. F:Xe Rat.).XA 8 0q= X [*306-1711] *306-25. F:X~8gYeRat.=. f'(X~sY).=-.X,YeRat [*306-12-22. *303-26. *306-14] Here X +8, Y must be taken in a sufficiently high type, otherwise X ~8Y may be null when X, Ye Rat. *306-3. F. (X/FL +8 V/P) +8 a-/T = X/p.o +8 (V/P A-8 a-/T) Dem. F. *306-14.:) F:, +a 0. P t 0. 'r tO). (X/,.t +8 V/P) +8; a-IT = (X X. p) +i-0(a xc0v)/(1 X. p) ~8aO-/r [*306-14] (Xx. Xpx xT)~0,.(F X. V X Cr) +,, (k e Xcp xla)'/(tk XOp X07) [*1 13-43] = [{X x. (p x0 r)} [U X,, ((V X,, T) +C (p X~ a~))}]/tb x 0 (p X,, Tr) [*306-14] = X//.L ~8 {(v X,, ') +C (x0 a-T)}/(p x0 r) [*306-1.4] = X/pi +8 (v/p +8, a-/, r)(1 F(1). *306-12.:) F. Prop 19-2

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292 QUANTITY [PART VI *306-31. -. (X~8 Y)+sZ=X+,g(Y+sZ) Dem. F.*306-3.)F: X=X//I.Y=v/p.Z=afrr-.. (X~8SY)~gZ = X+8 (Y~8SZ) (1 F..*306-25.) F: - (a[X, p1kvp, avrO) T. X= X/~k. Y= V/P..Z= CIT.). (X +8Y) +s Z=AX~S(Y +8Z)A (2) F. (1). (2). F. Prop *306-4. F.X/,x (V/P +8 cr/T) = (X>, X, V/P) +8 (X/P X, a-/T) Dern. F. *306-14.)F: X, vI,pV, 0, TE 7 NC ind., b4. v j 0.a- 0.)D. X/,a Xs (V/P +8 o-/T) =X/~4 X, {(V X0 7) +0 (p x, o-)I/(p x0 7r) [*305-14] = [X x0, {(v X0 T) +0 (p X0, a-)}]/Qj.k X0 p X,,T [*303-23] = [X x0,tt iX0 {I(v XT) +, (p X,-)fl/C(, XpX XT x) [* 13A43] = {(X xc /.x vX T') +c(X Xct IXcp xO-)I/(tX,a xp X~tk xT7) [*306-14] = (X Xc v)/(,a x, p) ~8 (X xc ar)/(t4Xc T) [*.305 1 4] = (X//it xs v/P) +8 (X/1 x8o0-T) (1) F. *305 2. *306 22.) F: ft! X/bt X, (V/P +, ojrT). X//.I, V/P, 0-/T e Rat. [*303-7].Hp (1) (2) F. *306,12. *305,143.) F: 4t! {(X/,i~t >X V/P) +8 (X/[L XS o-/T)j I. X/,a, VIP, o-/Tre Rat. [*303-7] D. Hp (1) (3) F:'-I Hp (1).. X/ IILX x(VIP ~8o-/T) =2A=(X/U X, V/P) +, (X/,u X, o/T) (4) F. (1).(4).) F. Prop *306-41. F. X x(Y+8 Z)= (X x8Y) +8(X x8Z) [*306-425. *305-2] *306-51. F.X+8 (V/1 X8X) = (V + 1)/i xXS Dem. F.*306-12. DF:.!X +8 (v/1 X.X).)D: X. v/1 xX eRat: [*305-3.*303-7]D:ve NCind: (21p, -).p,oE cNC ind. o- 0.X =p/a- (1) F. *305-2.)D F: J! (v ~+ 1)/i x8, X'}. D v+ 1)/i, X E Rat: [*303-7.*1 26-31]:v eNC ind: (gp, a).p, ceNC ind.- +0.X =p/o- (2) F. *305-142.)D F:v, p, a e NC ind. c =+ 0.). v/I x, p/o = (v x0, p)o-. [*3-06-13]:). p/ar +8 (v/i x, p/a~) = {p ~, (v x, p)jj/ o [*113-671] = {(v +C 1) X,,PI}/c [*305-14] (vo1)/i X, p/o (3) F. (1). (2). (.3).:) F. Prop

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SECTION A] SECTION A] ADDITION OF SIMPLE RATIOS29 293 *306-52. F:.X<rY. =E:XeRat:(EIZ).ZERat-t'0q.X4-gZ= y Dem. F. *306-13. *119-34.) F:aL v) p, o- e NC'nd. v +0. ot+0. X = jk/v. Y= p/CT. AL x,,, a- < vx, Xp. ~=(V x.p)-0(AL x a'). Z= ~/(V X,,0).:).X+8Z=(v x~p)/(V X,,a-) [*303-23] = P/CT [Hp] (1) F.(l).*30 4-113.) F:.X <rY.):Xe Rat: (2Z). ZeRat - t'q. X sZ = Y (2) F. *306 14.)D F: AL, v, p,

cTENC nd. v # 0. p j=O. CT~+0. X=AL/v. Z = p/a. Y =X+8Z.).D Y=1(A Xe CT) +c,(7vX. p)J/(V X,0CT) -[{(ALX, xC)+ (V X,,p)} x0'v]> 1-kxe(I'xe0a).[*304-1]).-X <r Y (3) F.(3).*304-1.:)F:XecRat.ZeRat-t'Oq.X+SZ=Y.:).X<ry (4) F. (2).(4). -) F. Prop The above proposition requires that X and Y should be taken in a sufficiently high type, namely at least in a type in which, if X = AL/v and Y = p/CT, where AL Prm v and p Prm CT, (v x,,p) +, 1 and (AL x,, CT) +0 are not null. Otherwise there may be no Z such that X +, Z = Y. *306-53. F:.AL,vENCind.v#0.CT=0. fl#0.): AL/V +S p/CT = AL/V +S a/fl. p/CT Dem. F. *306-12. D Hp AL/kv +8 p/CT =AL/v ~8 4/it. (p, CT E NC ind). D. AL/'V+s8 A=AP/CT. 1 [*306-25] Ip~ A/v, ~/, tj e Rat}. [Hp.*303-7]) — (~,q is NC nd). [*303.11.(1)]). q = p/CT (2) F.*306-25.) F: Hp. AL/v +8 p/CT = AL/v +, ~1. p, CT NC nd.. ~, f7e NC iLd (3) F. (3). *306-14. *303-39..) F: Hp (3).). {. x0C) +C (v x0p)} X0v Z'x,.= t(AL X0q) +- (v xe)1 XC V Xe CT [*113-43]:). (/ 6 CT x0 Xv xV Xn) +C (V2 Xe p XC 77) =(AL X CT7 Xe V XC 87) +0 (v,2 Xe 4 X CT). [*126-4]. V2 XC(pXc,q) = V2 X,(~ Xa). [*303 39])D. p/CT = ~/'q (4) F.*306-1.)F:pC=/7)A/+pC~A/~t/I(6) F.(5). (6)). Prop

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 *306-54. F:.X,YeRat.:):X+8,Y=X+8Z.E. Y==Z Dem. F. *306-25.)FI:. Hp.): X+8YeRat: [*306-25]:):X+,Y=X+,Z.:).ZeRat (1) F.(1).*306-53. 303-7.) F. Prop *306-55. F:Y<rX.:).r-..(aZ).X+,Z=Y Dern. F *1 17-291. *304-1.) F: Hp.)e(X <, Y). [*306'52]:). c..qZ). ZERat-L'Oq. X+,Z=Y (1) F.*306,24. *304-1.:)F:Hp.:).,(X~sOq= Y) (2) F. *306-25. D F: Hp. X~,Z= Y.:). ZeRat (3) F.-(1). (2). (3).) F-.Prop The following propositions are concerned with the existence of X +, Y in definite types. It will be shown that if X, Y exist in a given type, X +, Y exists in the next type, i.e. if X ~,'aand Y ~ talt exist, then (X +~Y) ~ too'/. exists, where X, Y~ are rationals. *306-6. F:p, pe D'UAU 'U.). +p)At'ae DI' Un P16U Dem. F.*305-23. FHp qt <p.).1u p0<2P 1 Similarly F:Hp.p <t.).it+p p< 2s9+c (2) F..(1). (2). *116-72.)F. Prop *306-61. F:t,,v,peD'UnC PU.D).(/.I/v +splv) rit./iteRatdef Dem. [*390371]) (it/v +8 p/v) rn toe Rat def:) D Prop *306-62. F:P,v,pED'UA P"U.).(,/V~8p/p)Ait.,'ieRatdef Dem. F. *303 39. DF Hp.).- /v +8 p/p =,/v +,v/v (1) F.-(1).*3006 61.F. Prop *306-621. F:u-e NC iid. 0- -+ I< Dem. F. *116'301-311. P.0 00+,2 (1) F. *116-321V331.)F. 12 _C1 +0 1 21 (2) F. *1 17-55. *126-5.) F. 22 _ 2 ~e I ~2'3 F. *305-231.) F: Hp. a- > 1. m,2 _a- +', 1< 20r.) (o-+0I1)2_0(c+Cl1) +'OI<,2c+0(2 x,, -). F.(1). (2). (3). (4). Ind uct.) F. Prop

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 *306-622. F Cid-tto12=/2-(x01&~0 Dem. F. *305-231 Ftc:H.) m1)2c {2 x0,(g I-)} +0 =,2 (1.*1 13-43. *120-416. F:Hp..2 x0,(p 0)} +0,2 2 x0 (2) F. (1).(2). F F: Hp.)., 1)2+ (2 x0Ft>F2 +0I (3) F. (3). *119-32. DF. Prop *306-623. F:Ft4,v,pcNCind.v<Ft.p~Ft.). (Ftxct)~c,(pxcp)<214+o1 Dern. F.*120-429.)F: Hp.:). (,u x0Ft) ~0(v xCp),<t2~c

1c)2 [*120-429.*306(622]) ~x, I-k) +. (v x. p) < (2 X. Ft2) (2 x. Ft) +a 2 [*306-621.*126-51] <2I'+c1:) F. Prop *306-624. F:Ft,v,p,o-eNCind.v<Ft.p-<Ft.a,<t.:). (Ft xG a-) +0 (v xe p) < 2/A~o1 [*306-623] *306-63. F:F,,, I'AP.).(tv8/- t~F a e Dem. F. *306 i62.)DF: Hp. v=,a. (Ft/v +,p/o-) ~tOOFe Rat def () F. *306-624. *305-24. *303-71.) F: Hp.v<Ft.p,<Ft.o-~Ft.) D.(Ft/v+,p/o-)~t,,'FeRatdef (2) Similarly F:Hp. v < p -a <,p a- <Ftp.)..,'/v +,p/a-) ttoote Rat def (3) F:Hp. V<Ftt ca- ~,,F.)(Ft/vs p/c-) ~t.oFe Rat def (4) Similarly F.(1) (4). (5).) F: Hp o- < p.) D. (Ft/v +p/o-) ~t.fte Rat def (6) Similarly F: H1p. Ft~< c-.). @z/v+p/-) to tFe Rat def (7) F.(6).(7).)F.Prop The following propositions are immediate consequences of *306-63. *306-64. F:(tv 1'a pc)~ ERtdf ~'+ /r ~). a e *306-65. F:X, YeRat def.).(X +8Y) to"C E Rat def *306-66. F:XYe C'H.). (X +8Y) ~too'C"cC'X eC'H *306-67. F: X) Y e C,'H'.:). (X +s Y) ~ too'C"IC'X e C'H'

*307. GENERALIZED RATIOS. Summary of *307. In this number we introduce negative ratios. If X is a ratio, what would ordinarily be called - X is X Cnv. This may be seen as follows. Suppose we have RXS. We then have R (X I Cnv) S. Now if R and S are vectors which carry us in the same direction, R and S are vectors which carry us in opposite directions, i.e. their ratio is negative. Hence calling the class of negative ratios "Rat,," we may put *307-01. Rat, = Cnv"Rat Df The sum of "Rat" and "Ratn" we will call "Ratg," where "g" stands for " generalized." Thus we put *307-011. Ratg = Rat v Rat, Df If /I <r < p/lo, we have!(/v) I Cnv} (I Cnv; <,) {(p/) I Cnv}. Hence we put *307 02. <n= Cnv;<r Df *307 021. > = Cnv<n Df If X and Y are generalized ratios, we consider X less than Y if either X, Y are both positive and X <, Y, or X, Y are both negative and X >n Y, or X is negative and Y is positive or zero. Hence we put *307'03. <g = (>n) J (<,) W (Rat - t'Oq) T Rat Df On the analogy of <n and <g, we put *307'04. H,= Cnv;H Df *307-05. H Hg = Hn H' Df We prove in this number that if X is a ratio, X Cnv = CnvX, and Cnv'(X Cnv) = Cnv (*307-21-22). We prove also *307-25. F. C'H n C'H, = A We prove that 0q and oo0 are their own negatives, but are not the negatives of anything else (*307'26'27'31). We prove Nr'H, = Nr'H (*307-41) and Infin ax.). Hg e r (307'46). None of the propositions of this number offer any difficulty.

SECTION A] SECTION A] GENERALIZED RATIOS29 297 *307-01. Rat, = j CnvI"Rat Df *307-011. Ratg =Rat v Rat,,, Df *307-02. <,,. = ICnv; <r Df *307-021. > = Cnv'f<n Df *307-03. <g =(>n) " (<r) VJ(Ratn - t'rO) Rat Df *307-031. >g =Clv'r<g Df *307-04. Hn = ICnv;H Df *307-05. Hg = fin4.- H' Df *307-1. F:RB (X ICnv) S..RX [*71-7] *307411. F: R(I Cnv;X) S.ERXS [*307-1] *307412. F. XICnvjCnv=X [*307-1] *307-13. F: XJICnv =YJCnv.E.X=Y [*307-12] *307-14. F: Y= X IC'v. =. X=Y ICnv [*307-12] *307-15. F: ft! X ~ K E. ft ' 1 (X ICnv) r (CnV"Kc) [*307-1] *307416. F:.K=CnV""K.:):f!X~K.=.f'(XICnv)~,K [*307-15] *307-2. F. (,u,/v) ICnv = Cnv I(,tt/v) [*307-1. *303-19] *307-21. F:XeRatv

t'Qoq).XlCnv=CnvIX [*307-2.*30.3-767] *307-22. F:X ERatv t'oo q'.Cnv'(X Cnv) =X jCnv [*307-21]. *307-23. F. Cnv"C'H, = C'H, [*304-28. *303-13. *307-22] *307-24. F:,L v, p, aEU' U.pu Prm v. pPrm o. p > o. o 0.) ft! (p/u) 2-L(gjv) ICn v Dem. F.-*303-32.) F. llp.)(HP, Q). P, QERel numn. P,0 C Q,0. P (p/-) Q. [*303-21] D:(HP, Q).P, QeRelnjum. P,0 C Q,,0. f1jPa A QP [*300-3]): (HP, Q). P,Qe Rel num. f! PO'A QP.PP A QI=A [*303-21]): (HP, Q). P (P/u) Q. { P (H/v) Q}:.)D F. Prop *307-25. F. (J'H n C'H = A Dein. F. *307 24. *303 13. F:p, v, p, aeU'1U. pPrin v. pPrm o. D. /v +(p/u-) 1Cnv(1 F. *302-22. *303-211. *30427-28. D F: X, Ye CIH.)D. F. (1).(2).) F:X, YEG'H.). Xt YICnv:)F.Prop

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298 QUANTITY [PART VI *307-26. F.0qjCflV=Oq=CflVj0, Dern. I -.*307-2.)F.09j IClv =Cflv 09 (1) F.-*303,615. *307-1) DF:1? (Oj Cnv)S.= YR I CG'S. [*33-22] = 1R I C'S. [*303-15].R09S (2) F.(1). (2).:) F. Prop *307-27. F. 00 qjI Cnv = 009q= CnvI oo 9 [*307-26. *303-62] *307-3. F: XeC'H.). f! (Xj Cnv)r~Relnum [*304-5. *307-16.*300-4] *307-31. F:XeRat-ffoq.:).XlCnv+0q.XlCnvziooq, [*307-3. *304'53.*303-62] *307-4. F: XH,,Y.. - (X Cnv) H(Y I Cnv) [*150-41. (*307-04)] *307A41. F.Nr'Hn = Nr'H [*.307-13. (*307-04)] *307-42. F: Infin ax.) Nr'H, Nr'IIn = [*307-41. *304-33] *307-43. F: X e C'Hn.). ft! X Rel numn [*:307-3] *307-44. F.09,q 00 q cn[*307-31] *307-45. F. Nr'Hg =Nr'H flN- I -N'I [*:307-25 41. (*307-05)] *307-46. F: 1nfin ax.) D.ae [*307-45. *304-33] This proposition requires q -I-i -I- q = q, which is easily proved.

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*308. ADDITION OF GENERALIZED RATIOS. Stummary of *308. In this number we have to extend addition so as to include negative ratios as addenda, and for this purpose we have to define subtraction of simple ratios. This is defined as follows: A A *308-01. X-8 Y= RS (Z):X, Y,ZeRat: Z+ Y=X. RZS. v. Z+8X=Y.RZS} Df That is to say, if Y <,. X, X -s Y is the ratio which must be added to Y to give X, while if X <, Y, X X- Y is the negative of the ratio which must be added to X to give Y. Thus we have *30813. F:. Y<X.v. YeRat. Y=X:.X-,Y=(iZ)(Z+ Y=X) *308'14. F:. X<r Y.v.X Rat. Y=X:).X-,Y= {(Z)(Z+X= Y)} Cnv We have, of course, X-sq = X (*308'22), Oq -8X = X Cnv (*308-23), and X-sX=Oq (*308'12). Existence-theorems for X-s Y are closely analogous to those for X +, Y and X x, Y. Also we have *308'2.: X, Ye Rat. -. X -, Ye Rat, We define the sum of two generalized ratios by means of the sums and differences of simple ratios, as follows: *308'02. X+g Y=(X+, Y) v (X -, YCnv) w (Y- X Cnv) (X Cnv+,Y Cnv) Cnv Df Of the four relations which occur in the above definition, all but one must be null if neither X nor Y is Oq. Thus if X and Y are positive, X -s YI Cnv, Y-s X Cnv, and X I Cnv +s YI Cnv are null; if X is positive and Y negative, X+ Y, Y- X I Cnv, and X Cnv + Y I Cnv are null; if X and Y are both negative, X +8 Y, X -s Y I Cnv, and Y - X Cnv are null. If X is Oq and Y is positive, X+ Y= Y-,X ICnv. X-8 Y Cnv = (XCnv+ YI Cnv) Cnv=A. If both X and Y are Oq, all four relations are Oq.

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300 QUANTITY [PART VI Hence we find *308-32. I-:X, YeRat.D.X+gY=X+gY *308-321. I-:XERat. YeRat,.:).X+gY=X-,YlCnv *308-322. F: Ye Rat. X eRatn.).X +g Y= Y- X~Cnv *308-323. F: X, YE Ratn.:). X +g Y= (X ICnv-i-8 Y Cnv) I Cnv The existence-theorems for X ~g Y are closely analogous to those for X +8 Y, and the formal laws offer no difficulty. We have *308-52. F:.X,YERatg.:):X +gY=X~gZ.=-.Y=Z *308-54. F:X, YeRatg.).(gZ).ZERatg.X+gZ=Y *308-56. F:. X <gY.=: Xe Rat,,:(HZ). Z eRat - t0q. X +g Z Y *308172. F:(X+gZ)<g(X+gZ').-. XeRatg.Z<gZ' *308-01. X-sY=iRSt(aZ):X,YZeRat:Z+,Y=X.-RZS.v. Z~8X=Y.RZSI Df *308-02. X +g Y= (X +8 Y) tu (X -s, Y ICnv) %u (Y-, XI Cnv) w (XI Cnv 48 Y I Cnv)I Cnv Df 301.F: Y<rX:).X-s Y=RS{J(3Z). ZeRat.ZsY= X.1?ZS} Dem. F. *306-55.:) F: Hp.).,,(51Z). Z +,X= Y (1) F.(1).(*308'01).) F. Prop ^AA *308-11. F: X<r y.).X-,Y=RS{(a1Z).ZeRat.Z+sX= Y.BZS} Dem. F. *306-55.:) F: Hp.:).,,(HZ). Z~8 Y= X (1) F.-(1).(*308-01).:)F. Prop *308-12. F: XERat.X=Y.).X8 Y=Oq [*.306-54-24] *308-13. F:. Y<rX-V.YeRat. Y=X:).X-8Y=(?Z)(Z~8Y=X) Dem. F. *306-52-24.) F: Hp.)D.(HjZ). Z+8 Y= X.ZERat (1) F.*306-54. DF:Hp.Z+8Y=X. Z'+8Y=X.D.Z=Z' (2) F.(1).(2). *308K1-12.) F.Prop *308-14. F:.X<r Y-V.XERat. X=Y:).X-,Y=f(iZ)(Z+,X= Y)} Cnv [Proof as in *308-13] *308-15. F:-,(X, Ye Rat). D). X-8Y= A [(*308&01)] *308-16. F:X,Y,-Rat. Y+,Z=X.:).X-,Y=Z Dem. F.*306-55. *304-221.:)F:.Hp.:):Y<rX-V.YeRat.Y=X (1) F. (1). *308-13.:) F. Prop

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SECTION A] ADDITION OF GENERALIZED RATIOS 301 *308-17. F:X,YeRat. X~8Z=Y.D.X-,Y=ZIC nv [*306-55.*308-14] *308-18. F: Y <r X)X.X Ye Rat - t'Oq Dem. F.*306-52.:)F: Hp.). (HZ). Z cRat - tf0q. Y+8 Z=X (1) F. (1).*308-13.DF. Prop *308-19. F X <r Y.).X -sYERats - tOq Dern. F.*306-52.)F:Hp.).(HZ). ZeRat - t'0q X +8Z=Y y (1) F. (1). *308-14. D F. Prop *308-2. F: X, YERat.=. X -sYeRatg, [*308,1218-19-15] *308-21. F: X-8Y-(Y-,X)'Cnv=Cnv!(Y-,.X) Dem. F. *308-1.314.) D F:.X<rY.v.XE6Rat-t'Oq. X=Y:).X-,Y=(Y —8X)jCnv (1) F. *308-13-14. *307-12.) D F:. Y <rX.V. YecRat - t'0q. Y =X::). X-Y= (Y-8,X) ICnv (2) F.(1).(2).*304-221. DF: X, Ye Rat.D. X- Y=(Y-,X) ICnv (3) [*307-21.*308-2] =Cnv I(Y-s X) (4) F. (3). (4). *308-1-5. D F. Prop *308-22. F:Xe Rat.). X-,Oq X [*306-24. *308&13] *308-23. F:X E Rat.D.q -,X =XlCnv [*308&2122] *308-24. F:(/)< X/) /~ / (0p c(~X)/i ~p Dem. F.*3041.:)F:Hp.:).Xx,,p>o,ixv (1) F. *303-23. *306-13. (1).) F: Hp.). R X x. p) - (px. v)}/(p,txp) +sv/p [*303-23.* 19 34] = X/p (2) F. (1).(2). *308-16.) DF. Prop *308-241. F:(X/P.) <r(,v/P).:).X/,tt-gv/p =[{(,ax0 v) -c (X x0p)}/(~s x~p)JJCnv [*308&24-21] *308-25. F:Xp,~E'rP.vp~/.)(/-vp~o'e' Dem. F. *305-24.) D F: Hp.)D. f (X x,,p) -0,(,a x,,v)} I t'",a,(p x,,,p) nt', e D'U n 6IU (1) F.(1). *308-24. *304-28. D F. Prop

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302 QUANTITY [PART VI *30825QANIY1PRTV [*305-24. *308-241] *308-252. F: X, l, v, peDI'U nU'LT6.)(X/,u-, v/p) ~t,00'pe C'HIq [*308-25251P12] *308-26. F: X, Ye iRat. X ~ tI11'.L, VY~ t,,j/.4 E R'.). (X-, Y)~ t~lE C'Hg [*308-252. *304-28] *308-261. F:X, YEC'H'.). (X-8 Y) ~t.CG"C'X ECGHg [*308-26] *308-3. F: f! (X -,Y ICniv).:). X eRat. Y eRat,, [*308-1-5. *307-12] *308-301. F:f!(X1Cnv+, YlCnv).:). X, YERat,, [*306-12.*307-23-12] *308-31. F:ft!(X4+g Y).).X, YERatg [*.306-12. *308&3301.(*308&02)] *308-32. F:X,YeRat.D.X~gY=X+,Y Dem. F. *308-3301. *:307-25. (*308-02).. F: X, YeRat - tfq.D. X g Y= X +8Y (1) F. *306-24. *308'22-3301.) F: X eRat - tOq.a Y= Oq.). X+g Y X = X+8 Y (2) F. *306-24. *308-3301. D F: X= q Y= q.)X +g Y= q =X4s Y (3) F. (2). (3).)D F:. XERat.Y =Oq.v.Y eRat.X Oq:)X +g Y =X + Y (4) F. (1). (4).)F. Prop *308-321. F:XEcRat. Ye Ratn)X +gY = X -,Y Cnv [*306-12. *308-3301. *307-25. (*308-02)] *308-322. F: Ye Rat. XeRat,,.). X +gY= Y-,X Cnv [*306-12. *308-3301. *307-2-5. (*308-02)] *308-323. F: XYe Ratn.D. X+g Y= (X Cnv +,Y CDV) ICnv [*306-12. *308&3301. *307-25. (*308-02)] *308-33. F: X +q Ye Ratg. * X, Ye Ratg [*.306-22. *308-232-31] *308-4. F.X +q Y=Y+gX [*306-11.(*308&02)] *308A1l. F.X+g Y(XI Cnv+g YlCnv) CnV Dem. F. *307 12. *34-26. (*308-02).)D F.(X ICnv +g YJCnv) ICnv =(X ICtiv +YJ Cnv) ICnv tv(X ICniv-Y) Cnv v (YI Ctiv-X) ICnv w(X~., Y) [*308-21] = (X I Cnv v~ YJ Cnv) ICnv v (Y-, X I Cnv) (X -8 YI Cnv> v(X +. Y) [(*308-02)] = X + Y.:) F. Prop

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SECTION A] SECTION A]ADDITION OF GENERALIZED RATIOS30 303 *308-411. F. (X+g Y) Cnv =XlCuv ~g YlCnv [*308-41.*307-12] *308-412. F:XlCnv+g Y Cnv=ZlCnv.=-.X+gY=Z [*308-411. *307-13] *308-42. F:X,YERat.:).(X-sY)~gY=X Dem. F.*:'308-1232.*.306-24. ~:llp.X=Y.:.(X-8Y)~gY=X (1) F.*308-18-32.:)F:Hp. Y<rX.:).(X-sY)+gY=(X-8y)~8,y [*308&13] =X(2) F.*30.8-19-322.) FHp. X <rY.). (X-8 Y) +g Y=Y-8(X8 Y) Cnv [*308-21] -Y -8(Y -8X) (3) F.-*308-13.)I-: Hp (3).D. X~8(Y-, X) =Y. [*308-16-18] D. X=Y-8 (Y-8 X) (4) F. (3). (4). D)F:llp.-X<r Y.:).(X -sY) -g Y= X (5) F. (1). (2). (5). *304-221. D)F. Prop *308-43. F:X, YeRat.:).(X+, qY)-,Y=X Dem. F. *308-32. D F: Hp. D. X +g Y= X +g Y. [*3)08-16.*306-22]:).(X +g Y) -8 Y =X: D F. Prop *308-44. F:.X,ZeRat.):X-,8Z= Y-,Z.=.X= Y Dem. F. *308-13-1415.)DF: X =Y.). X -,Z= Y-, Z (1) F. *308-2.)FHp. X -sZ= Y-s Z.:). Ye Rat. [*:308A42])(Y-8 Z)~+8Z= Y. [Hp]).(X —8 Z) ~8 Z =Y. [*308-42])X= Y (2) F. (1). (2). D F. Prop *308-45. F:.X,ZeRat.:):Z-8X=Z-8y.=.X=Y [*308-4421. *307-13] *308-46. F: X, YeRat. Y+0q)(X 8 Y)<g X Dem. F. *308-19.)F X <r Y. (X -. Y) e Rat, - t'Oq a X e Rat. [(*307-03)] D. (X SY) <gX (1) F. *308-12. F: Hp. X =Y.). X -Y= Oq [*304-46.(*307-03)] D. (X 8 Y) <g X (2) F. *308-13418. D F: Hp. Y <r X)(X -, Y) +8Y= X. X -8YeRat- t'Oq [*306-52] (X -(8 Y) <r X. [(*307-03)] (X. (X8 Y) <g X (3) F -(1). (2) -(3). DF. Prop

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304 QUANTITY [PART VI *308-47. F:XERat. YZeRat-ff09.:).X-8 Y#X+8Z Dem. F -.

*306-52 *308 46.) F: Hp. (X -, Y) < X+) [*304-201]).X-,Y+X+,Z:)F.Prop *308651. F:.XeRatg.):X+,gY=X.-. Y=Oq Dem. F.*308-33.:)F:.Hp.:):X+gY=X.:). YERatg (1) F.*30832.:)F:XeRat. Y=Oq.:).X+gY~X+,Y [*306-24] -x (2) F.*308-322.:)I:XeRat,. Y=Oq.).X+gY=Y-sX Cnv [*308-23.*307-12] -x (13) F. (2). (3).:) I-:. Hp.): Y=Oq.). X+g Y=X (4) F.*308-32.:)I:X, YERat.X+gY=X.:).X~ Y=X. [*306-24-54] D. Y= Oq (5) F.*308-321.)DF: X eRat. Ye Rat,,.X +gY= X.:).-X -,Y Cnv X. [*308-22-4l]:). Y Cnv =Oq. [*307-2] D. Y= Oq (6) F. *:308-322.) F: Xe Rat,,. Ye Rat. X ~g Y= X.:). Y-8s X Cnv X [*308-23.*307-12] =0Oq -8 X ICiiv. [*308-44] D.Y=Oq (7) F. *308-323. *307-14.:) F: X)Ye Rat,,.X +gY= X.:). XICnv +,YJ Cnv =X ICnv. [(5).*307 26]:). Y=0q (8) F. (1). (5). (6).(7). (8). DF:. Hp.D: X +gY= X.D. Y= Oq (9) F. (4). (9). DF. Prop *308-52. F:.X,YERatg.:):X+gY=X +gZ.E. Y=Z Dem. F.*308-321-47.:)F:X,YeRat. Y+q.XtgY=X+gZ.).ZEeRatn, (1) F. *308-51.:)F:XeRatg Y~Oq.X+gy=X+gZ:).Z=Oq (2) F.(1).(2).*308-33.:)F:X,YeRat. X+gY=X+gZ.).Ze6Rat (3) F. (3). *308-32. F:X, YeRat.X+g Y=X+gZ.).X+ Y=X +sZ. [*306-54]:).Y= Z (4) F.(4).*308-323.*307-13.:)F:X,YeRat...X+gY=X+gZ.:). YZ (5) F. *308-321-32-47.) F:X cRat. Y eRat,,. X+g Y =X+g Z.)Z.Z,,ERat - t'q (6) ZY F. (2) Y, Z Transp. D F: X cRat. Ye Rat, - t'Oq.aX +gY= X +gZ)Z+ 02 (7)

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SECTION A] SECTION A]ADDITION OF GENERALIZED RATOS30 3 03" F. (6).-(7).- *308-33.) F:X eRat. Y eRat,,- t'0q.aX +gY = X +Z.).ZecRat, (8) I- (8). *308&321.:) I-: Hp (8). X -, Y I Cnv =X - Z I Cnv. [*308-45.*307-13]).Y= Z (9) I - (9). *308-411. *307-13.) -: X cRat,,. Ye Rat. X+g Y= X+gZ.:). Y=Z (10) F. (4). (5). (9). (1O).) F: Hp. X+g Y= X+g Z.). Y=Z (11) F.-(I11).(*30 802).:)F.Prop *308-53. F:X, YERatg.:).X+g(Y+gqXlCnv)=Y Dem. F.*308&321.*307-12.:)F:X, Y~ERat.:).X+g(Y+gXlCnv)=X+g(Y-,X) [*308-442] =Y (1) F. *308-32. D F Xe Rat~t. YE Rat.)D. X ~g (Y+g X Cnv) =X ~g (Y+ XjI Cnv) [*308-4321.*306-22] = (Y+, XJ Cnv) -8X Cnv [*308-43-32] =Y (2) F. *308-323. *307-12.)D F::XeRat. YE Rat,,.:).X+g (Y+g X Cnv) =X -q,(Y Cnv +, X)jCnv [*308&.321.*306-22] = X -,(Yj Cnv +,X) [*308,17.*307-12] = Y (3) F. (t).) X, YERat,,.).X Cnv -g,(Y ICnv +g X'Cnv ICnv) =YjICnv. [*308-411])X Cnv + (Y+g X ICnv) ICnv =YICnv. [*308-412])X +g (Y+g X ICnv) =Y (4) F.(1).(2). (3). (4).) F.Prop *308-54. F:X, YERatg.:).(HZ).ZeRatg.X+gZ=Y [*30853-33] *308-55. F:.X,YZ,ZRatg.:):X+gZ=Y. =-.X=YtqZlCnv Dern. F.*308&53-52-4.)F:Hp.X~gZ=Y.:).Y+gZlCnv=X (1) F. *308-53,4.:)F:Hp. Y+gZICnv=X.:).X+gZ=Y (2) F. (1). (2).)DF. Prop *308-56. F:.X<g Y. =:XERatg:(aZ).ZeRat-t'0q.X+gZ= Y Dern. F. *306-52. *308-32.) F:.X<,Y.=-: XeRat:(HZ).ZeRat-t'0q.X+gZ=Y: (1) [*306-52-25] D): Ye Rat:(HZ).Ze Rat - t'0q wX+g Z= Y (2) YJ Cnv, XiCnv F. (2) X F:. X >, Y.) XeRat,: (aZ). ZeRat-tfOq. YJ Cnv+gZ=XjCnv: [*308&55-412]) X E Rat,,: (HZ). ZE Rat - t6'Oq. X +gq Z= Y (3) R. &W. III. 20

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306 QUANTITY [PART VI F.*308&32-53.*306-23. -: Xe Rat,. Ye Rat. D. Y +g X I

Cnve Rat - L'0q. X +g (Y +g X IjCnv) = Y (4).(1). (2). (3). (4). (*307 03). D -:.X <gY.): Xe Ratg: (3Z). Ze Rat - v'0q.X+g Z= Y (5) F *35103. (*307'03).:): X e Rat,, - t'Oq Ye Rat. D. X <g Y (6) I-. *308-55-412.) I-: X, Ye Ratn. Z e Rat - t'Oq X +g Z = Y. D. X I Cnv = Y I Cnv +, Z. [*306-52]).X >nY (7) F.(6).(7). D I:.X eRat,,: (Z). ZE Rat - tffOq. X +g Z = Y: D). X <g Y (8) F.(1).(8). F:. X eRatg: (gZ).- Ze Rat - t'Oq. X +g Z = Y: D. X <g Y (9) F. (5).(9). DF. Prop *308-561. F:. X <gY.: Ye Ratg: (:J). Z)e Rat - t'Oq. X +g Z= Y [*308-56-33] *308-57. F:X<gY. =-.XeRatg. Y+gX Cnv eRatt 'Oq. D.YERatg. Y+gXjCnveRat t'IO, Dem. F.*308-5550-6 4.) F:. X <g Y.-: Xe Rat,,: (Z). Ze Rat - t'Oq. Z = Y +g X (Cnv (1) F. *30855-561-4.) F:. X <gY. =-: Ye Ratg:(H[Z). Ze Rat - t'0q.Z= Y + X Cnv (2) F. (1). (2). DF. Prop *308-6. F: X, Y, Ze Rat.:). (X +gY) +g Z= X +g(Y+g Z) [*30832. *306-22-31] *308-601. F: X, Y,Ze Rat,..(X +gY) +~gZ= X +g (Y+gZ) Dem. F. *308&323. *307-12 F: Hp.:. (X ~g Y) +g Z= (X I Cnv +, YJ Cnv) I Cnv ~g (Z I Cnv) J Cnv [*308-411] = t(X I Cnv +, YI Cnv) +g ZI Cnvj I Cnv [*308-6. *306-22] = {X I Cnv +g (Y Cnv +g Z I Cnv)} I Cnv [*308-411] = X +g(Y ICnv +g Z I Cnv) I Cnv [*308&323] = X +g (Y+gZ)::) F. Prop *308&602. F:XA, v,p, TreNC ind. A, p, 7re^ef'O.). (X/lC +s v/p) -, 8// = (X//L 8, u/T) +g VIP Dem. F. *308-24. F: Hp. o/r <, X/Az.)-. (X/,~ +g VIP) -8 0/Tr= {(X xe P XO 7) +,D (A X Iv XO T) -0 (1t Xc P X. oj)I/(,t Xc P x, r)T. (X/,o -,o/T) +8V/pI= {(X x,, P X,,r) -0Q (P X, P X 0) +~ (1kX 7 X0r)I/(P XCP XO T) (1)

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SECTION A] SECTION A]ADDITION OF GENERALIZED RATIOS30 307 F. *308-241.:) F: Hp. X/p +8 v/p <r u/'i-.:). (X/1t +8v/p)-8 o/,r - [(ki x p X xo-(Xe XP xe T) -(,a Xe v xe r)}/(1 x. p x,'r)]J Cnv. (A/-k ajr)1- +g v/p = [{(/h X0 r) -8 (X X. ojj/(1kX.t x I Iz) Cnv +g v/p [*308-322-21] =[{(I-sxcpx~o-)-(AX")xcpxr)-(gx,,vx,,)}/(/Ixp x0'r)]jCnv (2) F *308-24-241.) F: Hp.X>/pu < al/r. 0/Tr<r X/p(I t~/. D. (X/IL cY/T)- +g V/P= [tu X. a) -c (X xc, Ir)}/Qi x, -r)] ICnv +,, v/P [*308-322-21> t(X x>%p x,,r)~0,,(j xc v xc r) -c (p X xP Xc, ac)}/(I Xcp X,,r) (3) F*308-16K12. F: Hp. X a-r)(x/k,t+v/p) -8lr = VIP=(X/ltk-,O-/)~ v/ V (4) F.*308-125317.) F: Hp. X/g t I-/P = 'T/r.)(X/1 +s V/P) s 0/TrOq(X/.8/r) +g V/p (5) F. (1). (2). (3). (4). (5).)F. Prop *308-61. F:X, YZeRat.:).(X+gY)-sZ=(X-8Z)~ gy [*308-602-32] *308,62. F: X, YeRat.ZeRat~.).(X+g Y)+gZ=X+g(y+gZ) Dem. F. *308&33321 F): lip. (X+g Y) +gZ =(X +g y) -Zq Cnv [*308-4] = (Y +9X) Z~ Cnv [*308&61] =(YsZ ICnv)+g X [*308-4] =X +g (Ya~ZfI Cnv) [*308&321]- X +g(Y+g Z):)F.Prop *308-621. F: X, YeRat,.ZeRat.).(X+g Y)+gZ=X+g(y+gZ) Dem. F.*308 62.) F:Hlp.).(X Cnv +g YlCnv) +gZj (nv=X Cnv +g(Yl Cnv +gZl Cnv). [*308-411]:).(X~gy)JCflv +gZIClv=XlCnv+g(y+gZ)lCnv [*308-411] = {X+g (Y+g Z~I} Cnv. [*308-412]). (X +g Y) +g Z =X + (y+g Z) F). Prop *308'63. F.(X +gY) +gZ=X +g (y+g Z) Dem. F. *308-6601P62621.) F: X, Y, ZcRatg..). (X+g Y)-fgZ=X+g(y+gZ) (1) F. *308-31'33.) F: -(X, Y, ZERatg).).(X +g Y) +g ZA. X+g (Y g Z> (2) F.(1).(2).) F. Prop 20-2

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308 QUANTITY [PART VI *308-71. I: XcRatg. Z<g Z'.). (X +gZ) <g (X+gZ') Demn. F. *308&57.) F: Hp.:). Z'+, ~I~Z Cnv e Rat -tfOq. [*308-56]). (X +qZ) <g (X +g Z)+g (Z'~g Z ICflV). [*308-63-53] D. (X +g Z) <g (X +g Z): D F-. Prop *308-72. F:(X+gqZ)<g(X+gZ').-=.XeRatg.Z<,qZ' Dem. F.-*308-33. F: (X +gZ) <4 (X +gZ'). D. XZ,Z7'e Rat, (1) F.-*308-57.) F:(X+qZ)<g(X+gZ').:).f(X+gZ')~g(X~ Z)ICnvlERat-t'0q. [*308&411-63-53].(Z' +g ZjI Cn v) e Rat - vLoq (2) F.(1).(2). *308-57.:)F:(X+gZ)<q,(X+gZ'r).).Z<gZ' (3) F.(I).-(3).-*308-71. D)F.Prop *308-8. F: X, Ye Ratg X x Ato~/~, Y~ ti11'/L C C'Ig.:).(X~+ Y) t.; c C'Hg [*308-32321P322-323. *306-64. *308-26] *308-81. F:X, YeC'Hg.).(X +gY) ~t~o'C"C'X c C'Hg [*.308&8]

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*309. MULTIPLICATION OF GENERALIZED RATIOS. Summary of *309. The subject of this number is simpler than that of *308, because it requires nothing analogous to the consideration of subtraction. The product of two generalized ratios is defined as follows: *309 01. X xg Y=(X x, Y) w (X Cnv x8 YI Cnv) (X x YI Cnv) Cnv (X Cnv xY) Cnv Df As in *308, three of the four products concerned in this definition will be null in any given case (unless X=Oq or Y=0q). Hence *309'14. F: X, Ye Rat. D. X x, Y X xs Y *309'141. F: X e Rat. Ye Rat. 2. X x Y= (X x, Yi Cnv) j Cnv *309142. F: YeRat. X eRatn.:. X xg Y=(X Cnv x, Y) Cnv *309'143. F: X, Ye Rat,. D. X x, Y= X Cnv x, YI Cnv The propositions of this number are merely generalizations of those of *305. The proofs of the formal laws are straightforward, but the proof of the distributive law (*309'37) is long, because of the multiplicity of different cases. *309'01. X xg Y= (X xs Y) w (X Cnv x, YI Cnv) w (X xs Yi Cnv) I Cnv v (X I Cnv x, Y) I Cnv Df *309'1. F. X x, Y=(X x, Y) w (X j Cnv x, YI Cnv) w (X x, YI Cnv) | Cnv w (X Cnv xs Y) Cnv [(*309'01)] *309-101. F: X e Rat - 'Oq.. X Cnv x, Y= A [*3052. *307'25] *309-102. F: X e Rat, - L'Oq.). X X8 Y= A [*05'2 *307-25] *309'11. F:! X xg Y.. X, Ye Ratg [*305'2. *309-1] *309-12.. X x Y== x X [*305'11. *309-1] *309'121... X xg Y= X] Cnv x, YI Cnv = (X xg YI Cnv) | Cnv = (X Cnv x, Y) Cnv [*309'1. *307'12]

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310 QUANTITY [PART VI *309-122. F.Xxg YjCnv=XjCnvxg Y=(X xgY)ICnv [*309-121. *307-12] *309-13. F:XYeRat-tfOq.:).Xxg Y=X x8Y [*309-1101'12] *309-131. F:. X =Oq. Ye Rat - t'Oqav. Y= Oq.-X eRat -t'Oq:). X xg Y= X XsY= Oq Dem. F. *309-101.:) F:X=0O..Ye Rat-t'Oq:).X X.~ Y= (X xY) v(XI Cnv xY) ICnv. [*307-26. *305-22] D. X x9 Y= X iY= Oq (1) F.(1).*309-12.:)F: Y=q. X cRat-t'0q,.). X g Y X 8Y Oq (2) F. (1).(2).) F. Prop *309-133. F:X=Oq Y~Oqo.).XXgY~XXgY=Oq [*309-1. *307-26. *305-22] *309-14. I-:X,YeRat.:).Xx Y=X x8Y [*309-13-131-133] *309-141. F:XeRat.YeRatn,.:).XxgY=(XxYlCnv)ICnv [*309-121-14] *309-142. F: YeRat.XeRatn,.:).XxgY=(XICnvxY)JCnv [*309-141-12] *309-143. F: XYeRat,.). Xxg Y= X Cnv x8 Y!Cnv [*309-14-121] *309-15. F: X, Ye Ratg. E. X Xg Ye Ratg Dem. F. *305-3. *309-14'143. F:. XYe Rat. v. X,Ye Rat,,:). X xg Ye

Rat F. *305-3. *309 141-142.) F:. Xe Rat. Ye Rat,,. v. X eRat,. Ye Rat:)D. X xgYe Ratn F. *303-72. (*307-01-011). F::X xg Ye Ratg. D. f! X xgY F. (4). *309-11.)F: X g Ye Rat,.:). X, Ye Ratg F.-(3). (5).) F.Prop *309-16. F. (Xxg Y) xgZ=X xg (YxgZ) [*305-41.*309-1] (1) (2) (3) (4) (5) *309-17. F:X, Yeet'0qviAloq~D.xg Y=Cnv'(Xx Y) Dem. F. *309-1.:)F X Xx9 Y=(Xx, Y)W(XjICnv X.Y ICnv) ti(XxYlCnv) ICnvv,(X!Cnv xY)ICnv (1)

SECTION A] MULTIPLICATION OF GENERALIZED RATIOS 311 F.*305'12. D-: llp.:). X xY=Cnv'(X xY) (2) F. *307-22. D-: Xe Rat.D. X ICnv =Cnv'(X ICnv) (3) F (3). D F: Ze Rat. X =ZICnv.). XI Cnv =(Zi Ctiv) ICnv [*307-12]= [*307-14] = Cnv'(X Icnv) (4) F.(3).(4). F: X eRat. D. X ICnv =Cnv'(X ICnv) (5) F.(2). (5).:)I-: Hp. X, YE Ratg,*. XjI Cnv x, Y I Cnv = Cnv'(X ICnv x8, Yl Cnv).Xs x YCnv =Cnv'(X x8, Y ICnv). X ICnv x8, Y== Cnv'(X ICuiv X8 Y) (6) F(1). (2).(6). *309-1. DF: Hp. X,Ye Rat,,. D. X gY= Cnv'(X xgY) (7).*303-13-7. D X, Ye Ratg- t'0q..XYe Rat, t'Oq (8) F-. (8). *309-11.) F r(X, Ye Ratg v t'oo9)..Xx gY=A. Cnv'(X x,,Y)=A (9) F.(7). (9).:)F. Prop *309-21. F:.X,YeRatg:X=0qv.Y=0q..=.XXgY=0q Dem. F. *309-14-141. *305-22.*307-26.:)F: XeRate Y~Oq.:).XXgY~Oq (1) F.*309-15.:)F: X gY= Oq X).X Ye Ratg (2) F. (2). *309-14-141-142-143. *:307-26.-) F:. X gY= Oq.:X Xs Y Oq v..X IClv xY) Clv =Oq. v. X x, YJ Cnv = q v V. XjI Cnv x, Y= Oq: [*305-22.*307-26]): X = Oq' av. Y= 0q2 (3) F. (1).(2). (3).:)F. Prop *309 22. F: X, Ye Ratge ~t'Oq. X x Ye Ratg - t'Oq [*309-21. Transp] *309-23. F:X eRat g-t'Oq)X gX=1/1 Dem. F. *309-13. F FXe Rat - t'0q.0 XgX= X sX X [*305-52] =1/1(1 F. *309-121. *307T22.:)F Ye Rat - t'q aX = YlCnv ~XgX = Y xgY [(1) = 1/1 (2) F.(1). (2).:)F.Prop *309-24. F:XeRatg,).Xx 1/1~X Dem. F. *309-14.:)F:XeRat.:).Xx 1/1=Xxs1/1 [*305-51] =X(1) F. (1).*309-142.:)F:X eRat,,.).X xg1/1= (X Cnv) Cnv [*307-12] =X(2) F. (1).(2).) F. Prop

312 ' QUANTITY [PART VI *3Q09-25. F-:.XAERatgA+0q,.):XxgA=A'.-.X=A' x A. Dem. F. *309-2324-16. DF: Hp.D.X=Xx A AxgA F.(I X2- a30-1 -DF: Hp.X). A =A'.) xg = x =A' x9A (1) (2) (3) 1-.(3).)~DF: Hp. X =A'x, A.D. X xA =A' F. (2). (4).:) F-. Prop *309-251. F:.X,A'eRatg.A+0q.):XxgA=A'E.X=A' xgA [*309-25-15] *309-26. F:X,YeRatg.XtOq.:).(HIZ).ZERatg.XxqZ Y Dem. F.*309-25.:)F: HpZ= Yx X.). ZxX= Y F. (1). *30915 512. D F. Prop (4) (1) *309-31. F:X,YeRat.ZeRatg.:).(X+g Y)xgZ=(XXgZ)+g(YXgZ) Dem. F. *308&32. *309 14.) F: Hp. Ze Rat.:).(X+gY) xgZ=(X+,gY) x8Z. XXgZ=XX8 Z. YXgZ= YX8Z. [*306-41]:).(X+g Y) XgZ=(X XgZ) +g(Y XgZ) F.*309-122.:) F: Hp. We Rat.Z= WICnv.:).(X+gY)XgZ=I(X+g Y) xgW} ICflv U(1) = {(X Xg -W) +g (Y xg -W)} ICnv [*308-411.*309-122] = (X xg Z) +9 (Y Xg Z) F. (1). (2).:) F. Prop (1) (2) *309-311. F:XYeRat,,.ZeRat, *).(X+gY)XZ= (XXgZ)+g(YXgZ) Dem. F. *308'41. *309 12 2. F: Hp.). (X +g Y) xgZ ={(X I Clv +g, Y I Cnv) xgZJ I Cnv [*309-31] = {(X I Cnv xg Z) +,, (Y ICnv xg, Z)} I Cnv [*309-122.*308-41] = (X Xg Z) +g (Y xg Z):) F. Prop *309'32. F:(v/p) <r (X/p).

ojr e Rat.:). Dem. ~~(X//.48i v/p) x g cr/T= {(X xp) -,O(P'. xc v)) xe o}/(tk xcp X.'r) F. (1). *309-14. *305-142.:) F. Prop

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SECTION A] MULTIPLICATION OF GENERALIZE~D RATIOS33 313 *309-33. F: X/ pa, v/p, a'/'J- c Rat.). (X/,a -g v/p) xg (air7) = (X/p xg o-/,r) -sg (v/p xg o(7/7) Dem. F. *309-14. F: Hp).X/, xgajo-/7-X/p x,,ojr. vIp x, o/7-= v/p x, oj'7 [*305-142]) X/,tt xg ai =7(x0, a-)/(pi X,,r). v/ XIg xair = (v x0, o7)/(p x,, r) (1 F (1). *308 24.) F Hp.(V/P) <r (X/jLL). (X/1t Xg ai/r) 8,(v/p X, o/7r) {(X XO a) Xc (p Xe T) -0(P X.) X. (vX xaj}I/Qp X.p X. T2) [*303-38] {(X X 0a xp>() Xv X,, CJ/(~txp Xc Xr) [*309-32] (X/P -8 v/p) XgojrT (2) (V/P Xg, 0air) - (X,/,ttX,~ Cr/) =(V/P -s X/g) X, alT/ [*308-21.*309'122] X// Xg u/) -g (V/p X g a'/r) =(X/p. -s V/p) X g u-/ (3) F.*308-12.-*309 21.) F: Hp. X/pc = v/p. (X/)M -s V/P) X g o-/Tr- 0q2. (X/,ct X g a1r) -8 (V/pX xg u'/) = Oq(4) F.(2). (3).(4).) F.Prop *309-34. F: X, Y, ZEcRat.). (X -5 Y) x Z= (X XgZ) -s(Y XZ) [*309-33] *309-35. F:XZeRat. YERatn.:). (X+gY) XgZ=(XXgZ)+g(yXgZ) Dem. F. *308-321.)F:Hp.).X +,Y= X-8 YJCnv. (XXqZ)+g(YxgZ)(XxgZ)s(YlCflvxgZ) (1) F.(1). *309'34.) F. Prop *309-36. F:X, ZeRat, Y eRat)(X +g Y)Xg Z= (X X Z) +9g(Y XgZ) Dem. F. *308A41. *309,121) F: Hp.). X+g Y =(XjICnv+g Y ICnv)JICnv.X x gZ =X ICnv xg ZICnv. Y XgZ= YJ Clv xgZl Cflv [*309-122])D. (X+g Y) xgZ=(X I Cnv +g YJ Cnv) xgZJ Criv. (XXgZ)~g (YxgZ>=(XIClv xgZlCnv)-ig(Yj~nv xgZlCnlv) (1) F.(1).*309-35.:)F. Prop *309-361. F:X eRatg.Ye Rat,,.Ze Rat.. (X+g Y) xZ=(X xgZ)~g(YxgZ) [*309-311P36]

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314 QUANTITY [PART VI *309-362. F: X,Ze Ratg. YeRatn.).(X+gY)xZ=(X gZ)+, (YxgZ) Dem. F-. *309122. *308-41. F. (X +g Y) Xg Z = t(X +g Y) xg Z I Cnvj I Cnv. (X xgZ) +, (Y xg Z) = (X xg ZCnv) +g (Y xg ZI Cnv)jCv (1) F. *309-361.) F: Hp. Z e Rat,.. (X +g Y) x, Z I Cnv =(X x Z Cnv) +g(Y x, ZICnv) (2) F. (1).(2). D F: Hp.Ze Rat,. D. (X +g Y) xg Z = (X xg Z) +g (Y xg Z) (3) F.(3).*309-361. F. Prop *309-363. F: X, Y,Z e Ratg..(X +gY) xg Z= (X xg Z) +g(Y xg Z) Dem. F *. 309 35 12.2 *308-4.) F: Y, Z e Rat. Xc Rat,.. (X +g Y) xg Z = (X xg Z) +g(Y xg Z) (1) F. *309936. D F: Ye Rat. X, Z e Rat.. D. (X +g Y) xg Z = (X xg Z) +g (Y xg Z) (2) F: Xe Rat,. Ye Rat. Z e Ratg. D. (X +g Y) xg Z = (X xg Z) +g (Y xg Z) (3) F.(3). *309-31.) F: X e Ratg. YERat.Ze Ratg.:). (X +9 Y) xg Z = (X xg Z)~+g (Y xg Z) (4) F.(4). *309-362. D F. Prop *309-37. F.(X+gY)XgZ=(XXgZ)+g(YXgZ) [*309-363-1115. *308&31P33] *309-41. F:.A eRat - t'Oq.aD:(A xg X) <g Y..X <g(Y Xg A) Dem. F. *30856.F F:. (A xg X) <g Y. A xgXeRatg:(aZ).ZeRatt 'Oq.(A XgX)+gZ= Y (1) F. (1). *309-15. D F:: Hp. D):. (A xg0X) <g Y. X e Ratg: (HZ). ZZe Rat - t'0q. (A Xg X) +g Z= Y: [*309-25-37-23-24])D: X e Ratg: (HZ). Ze Rat - t'Oq - X +g9(Z XgA> Yxg A: [*3531*309 13]): XEc Ratg: (aZ'). Z' e Rat - t'0,. X +gZ'= YxgA: [*308-56] D: X <g (Y x gA) (2) Similarly F:. Hp.):X<g(YxgA).).(A xgX)<gY (3) F. (2). (3). D F. Prop

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SECTION A] MULTIPLICATION OF GENERALIZED RATIOS 315 *309'42. F:. A e Rat, - 'Oq.: (A xg X) <g Y.. (Y x A) <g X Dem. F. *307'4. 309'122. F:.Hp.: (A xX) < Y. -. (Y ICnv) <g(A Cnv xgX). [*309-41.*30722] -. (Y C nv xg A I Cnv) <g X. [*309121]. (Y xg A) <g X:. F. Prop *309-5.: X, Ye Ratg. X ~ t,',u, Y C t, ' e C'Hg.). (X xg Y) C too, CeYg [*309-14-141'142-143. *305-26] *309'51. F: X, Ye C'Hg.. (X xg Y) too'C"C'X e C'Hg [*309-5]

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*310. THE SERIES OF REAL NUMBERS. Summary of *310. Real numbers, as opposed to ratios, are required primarily in order to obtain a Dedekindian series, so as to secure limits to sets of rationals having no rational limit. If rationals and irrationals are to form one series, it is necessary to give some definition of "rationals" other than "ratios," since the series of ratios (assuming the axiom of infinity) is not Dedekindian, and is not part of any arithmetically definable Dedekindian series. But in virtue of the propositions of *212, the series of segments of the series of ratios, i.e. the series;'H, is Dedekindian, and this series contains a series, namely H;H, which is ordinally similar to H. Thus the properties which we desire real numbers to have will result if we identify them* with segments of H, > — and give the name "rational real numbers" to segments of the form H'X, i.e. to segments which have ratios as limits. Thus H'X is the rational real number corresponding to the ratio X, and a real number in general is of the form H"X, where X is a class of ratios. H"X will be irrational when X has no limit in H. Since real numbers involve classes of ratios, the ratios concerned must be of some one type, and cannot be typically indefinite. Thus, as might be expected, hardly any of the properties of real numbers can be proved without assuming the axiom of infinity. In the present number, however, we shall be mainly concerned with just those few simple properties which are independent of the axiom of infinity. The series s'H, by which real numbers are to be defined, has both a beginning and an end, namely A and D'H (which = C'H if the axiom of infinity holds). D' will be infinity among real numbers. It is not convenient to include it in the series of real numbers as defined, just as it was not convenient to include ooq in the series H or H'. Again A is not naturally to be taken as the zero of real numbers, which should rather be taken as being t'Oq. Thus we are led to the two following definitions, in which 0 is the series of positive real numbers other than zero and infinity, * On this definition of real numbers, cf. Principles of Mathematics, Chap. xxxIIi.

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SECTION A] THE SERIES OF REAL NUMBERS 317 while O' is the series of zero and the positive real numbers other than infinity: *310'01. = (sH) (- 'A - I'D'H) Df

*310'011. O' = Lt'Oq * Df These notations are framed on the analogy of H and II', the letter O being chosen to suggest 0, the relation-number of the continuum. Although we do not have Nr'O= 0, we have Nr''H =, and therefore (*310'15) i - Nr' +- i = 0, and Nr'O' - i = 8 (assuming the axiom of infinity). Thus the relation-number of (is simply that of a 0 with the ends cut off. We put further, on the analogy of Hn, Hg, *310 02. n = (s'H,) C (- tA - t'D'H,) Df *310'021. O'n = Oq *< + On Df *31003. (9g= On ' Df Thus (, is the series of negative real numbers, O', the series of zero and the negative real numbers, O9 the series of negative and positive real numbers (infinity always excluded). The class of positive real numbers is C'O, of negative real numbers C',n, of all real numbers (excluding infinity) C'O v t'I'Oq v C'(O. If v is a positive real number, Cnv"v is the corresponding negative real number (*310'16). The properties of O, 3O,, Og in respect of limits, continuity, etc., result from the properties of 0 as proved in *275, and from the properties of series of segments as proved in *212. Instead of taking the series of segments as constituting the real numbers, it is possible to take the series of their relational sums, i.e..;O. This depends on the fact that h; OsmorO (*310'33). The chief advantage of ~;O is that it is of the same type as the series of ratios. We shall show in *314 how to construct the arithmetic of real numbers defined as the relational sums of segments; until then, we shall regard real numbers as segments of the series of ratios. *310'01. O = (s'H)) (- tA - t'D'H) Df *310-011. O'= t'Oq *- (H Df *310 02. ~,, = ('H,) = (- tA - t'D'H,,) Df *310-021..'n = LOq t+ On Df *31003. (3 = n, - 3' Df *310-1.. (3, O', n,, On, g9 e Ser [*304'23. *307-4125. *204-5. 212-31] *310'11.: LOV. -.I, e D'He - tIA - t'D'H., a C v. u = v. -., v D'H~. g!,. D'H - v. ' v -. -e./, v E D''H n 'H. (fe C v. C - v [*212-23'132. *211'61. (*310'01)]

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318 *310-111. QUANTITY [PART VI I- pO,. v e D'(Hi,), - t/A - t'D'IIH,. u C V.u / v. =.t, v e D4H e g! s.:fa! D'IH, - i.:f ~! v - pL. CIO. v' e I't~Oq V C'IOV v.~ -" tlOq. a v E CIO [(*.310-03)] *310-114. F:.I pH,i,'v. I=: /t = t6O q IV E C'en a Vs.n [(*310*021)] *310-12. F. C - = I rH II = D'HE6 - tIA - t/D'H. C'(,)n=D'Is'Hn A (II''H = D'(Hn)e - iA - iD,'Hn [*212-132] *310-121. F. IC'e C Cl ex'D'H. C'~n C Cl ex'D'H,, [*310-12] *310-122. F:2! 3. =!O A O= =A On~.EA!e'n f.t! eg [*212-14. *161-13. *304-27] *310-123. F: a! 3. CIO'= tl'lt'q U C'(E. = 6 t't''Oq V C'eO. G'eg9 - CIe_ V t'I/Oqu CIO' [*310'122. *1 61-14] *310-13. F. C'eA C'e -A. A - lCIOnA Dem. F.*310-11-111.) F: ek'eC.)V EC")n:.,C D'H.v~'C D'Hn.a!,t.! vI [*307-25]:),ul=v.,itiv=A:)F. Prop *310-131. F. t'Oqr~ E C'e vC'e [*304'282] *310-14. F. IO smorO [*212-72. *307-41] *310-15. F: Infin ax.:) e' (-F C'1H, e'n -1- CG'Hn, CG'H,,*F Og+-F C'He6 [*304-33. *310,14. *275'21] *310-151. F: Infin ax.:) I O') 0f E Ser A comp A semi Ded [*310-15. *275-1. *271-18. *214-74] *310-16. F: vE ceC6. =.I Cnv''v E "O [*310-12.I (*307-04)] *310-17. F. I Cnv"'I Cnv"lv= v [*307-12] *310-18. F: j Cnvllv. =- Cnv"~ [*310-17] *310-19. F: p= v. Cnv",/k = JCnv (v [*310-17] *310-31. F:, /e C'O V C'&,,. I D! (h",) ~ Re] nu m [*304-5. *310-121]

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SECTION A] SECTION A]THE SERIES OF REAL NUMBERS31 319 Dernz. F -.*310-31. *303-62.)D D. hc~~~~k + h~~~v (1) F. *310-12-31. *307 25) D F: geC'16. v eC'e,,,. D.hp+hv(2) F.*310-11.)- D:.,ev.): a! v- /I: Similarly F: Ft, v C'eO.. p +V. D. h'it# 'v (5) F. (). (). (4. (5) D F. Hp D +. D.hclz h~v(6) F. (6). Transp. DF. Prop *310-33. F.~;e smor)9;,smor),,.;Egsmor eg [*310-32]

*311. ADDITION OF CONCORDANT REAL NUMBERS. Summary of *311. We define a set of real numbers as concordant when all are positive or zero, or all are negative or zero, i.e. when all belong to C'O' or all belong to CO',e. Given two concordant real numbers fL and v, we define the sum of F and v as the class of sums, in the sense of *308, of a member of Fp and a member of v, i.e. as W {(3jM,N).Me t.Nev. W=M+gN}, i.e. as s' +~g"v, in virtue of *40'7. It is easy to prove that, assuming the axiom of infinity, the sum so defined has the properties we require of a sum. We denote the sum so defined by ",t +pv." In order to insure that Ft+p v shall be A unless FA, v are concordant real numbers, we put *311'02. pL +p v = X {concord (a, v). X e s', +,gJ})f Thus if at, v are concordant real numbers, p +p v = s'tI +g"r (*311'11); if not, p +p v = A (*3111). A definition of addition which applies to real numbers of opposite sign will be given in *312. The commutative and associative laws for +p (*311'12'121) follow at once from the dorresponding laws for +g. Assuming the axiom of infinity, we prove without much difficulty that the sum of two positive real numbers is a positive real number (*311'27), and the sum of two negative real numbers is a negative real number (*311'42). In these proofs, when propositions of previous numbers involving " Rat" are used, " Rat" is replaced by CGH' and "Rat - t'q" by C'H. This is legitimate in virtue of *304'49-34. In *311'511 we prove (assuming the axiom of infinity) that if 4 is a positive real number, and Y is any positive ratio, however small, there are members X of ~ such that Y+g X is not a member of A, i.e. given any positive real number, there are rationals differing from it by less than any assigned positive rational. This proposition is useful, and is used in proving that if &, V are positive rationals, each is less than +p/ (*311*52). The converse of this proposition, i.e. the proposition that, if pLOv, there is a positive real number

SECTION A] ADDITION OF CONCORDANT REAL NUMBERS32 321 X such that v = pt +p, X, is proved in *311-621P64, after a considerable amount of work. Thus we have *311-65. F:nla.:3.~ vC~(X.e' -~~ We have, of course, a corresponding proposition for R, (*311P66). From *311-65 we deduce without difficulty that if p is less than v (p, v being, positive real numbers), then X ~+pu is less than X +p, v (X being a positive real number), i.e. *311-73. F:Infin ax.XeG'eO. pcHv.)(+p,.t) e(+pv) whence (with the corresponding proposition for,,) we deduce *311175.

F:. Jnfinax.conicord(X,pt). '~/,X~v=.,~ which secures the uniqueness of subtraction. *311-O2. p~p= X concord (~,uv).X Es'1 — gv Df *311-1. F:-concord (F,v7).). upv = A [(*311P02)] *311-11. F:concord(,a,v).:). FLpS/U~" W{(aIM, N). Mep. N ev. WM+g NJ [(*311-02)] *311-12. F. FLp+p v= v +p/. [*311-1,11. *308-4] *311-121. F.('X +pb) ~pv X +p,(p+v) [*311-1-11.*308-63] *311-13. F: concord (pt, v). =-. concord (I Cnv"1s, I Cnvllv) *311-14. F concord(FL,IICnv"zv). =.concord (I Cnv"S,uv) [*311K13.*310-17] *311-15. F:concord (p, ICnv I v).) concord (,a, v) [*310-13-16] *311-2. F:Infin ax. ~ C G'H. X OR G'1 X +g""H"4 H"cX +g"~ ^ HIPX Dem. F.- *308-72.- *304-34-401.):. Hp. D: Y eX +g"H"4. (aZ, Z').- Z'e4.Z eU'll Y=X +g Z (X +gZ) H(X +g Z'). [*37-6] (H~Z, Y'). Z eC'H.Y= X+g Z.YPe X+g"~.YHY'. [*306-52] Ye YE H"X +g,',4. XHY:.) F. Prop *311-21. F: Jnfinax.~C,HH~X CH:.~XC 6X gc Dem. F.*306-52. *304-401.)3F:. Hp.)3: Ye ~-.) XH(X +g Y): [*40-51-61] D: X e H",X +g,,'4 F.(1).*304-23. DF.Prop R.&W. 1II. (1) 21

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[PART VI 322 QUANTITY *311-22. F:Jnfinax.4CC'fl.a1!~.XeC'H)'. H"cX~9"4 = H~'X wX +g"H"cc De~rn. 4 F.*304-23.) F. H"X gt - (H"cX+g'cc AH*'X) v(H"X +g"~nH'X)(1 F. (1).*311-221.) F. Prop *311-23. F:Jnfinax.~~eq'e.XeC'H.:).H"cX +g"c~=H*'X vX+g"H"~ [*311'22. *310-12] *311-24. F:. Infin ax.4~e C'e. Ye C'H.) Dem. F.*304-3l.:)F: Hp.:). (aW). We4K. WHY. [*306-52] D. (HZ, W). Wc~. ZHY. Y=:Z~gW::)F.Prop *311-25. F:Infiniax.~,nqEC'e.).C~tpfl.nfC~+P1) Dem. [*311P24] D). e 1 g~i' (1) F. (1). *311 11. D)F: Hp. D.iq C. +p, (2) F. (2). (3).) F. Prop *311-26. F:Infln ax. 4,eCIE).D.H"I(~ +pfl +pq Dem. F. *311P23.) F:. Hp.): Y cfl. D.H"(~ +g Y) H* Y v(H"c)~g Y: [*311-25.*310K12] ~ +pq:.) F. Prop Dem. II*311-26.*310-12]). c ClO v' jt'D'H(1 F. *310-12. *211l703.) F: Hp.). (3M, N). M, NEcD'H. Mcp'JI"~. N epH'H" [*308&32-72.*306-23] D. (SM, N). M +g Ne p'H`1(~ +p n) A D'H (2) F. (2). *2005. DF: Hp,. D +pn4+D'H (3) F. (1). (3).)D F. Prop The axiom of infinity is essential to the truth of the above proposition, for if it fails we have E! B'H. B`H` —e e,n whilea ~C'e.:). B'H ep.

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SECTION A] ADDITION OF CONCORDANT REAL NUMBERS32 323 *311-31. F. jCnv"l(,at+,i) = (jCnv",tt) j (I Cnvl,,v) Dem. F.*311'13-1. F:.concord (At'v).:). ICnv"'(At+pv) =A. (Cnv"AtL) +p(I Cnv"'v) =A (1) F. *311-13-11.. F:concord (pt, P).).- Cnv"(at-i,)= Cnv cs't +g"9C V [*308-411] s'(j Cnv"Atu) +g"'(j Cnvf~v) (2) F.-(1).-(2).)F. Prop *311t32. F.- Cnv""(/z +p Cnvllv)=(Cnv"Atr) +Pv [*311P31. *310-17] *311-33. F. A -pnv'= Cnv"{I(Cnvl",a) +p(I Cnvllv) [*311P31.*310-18] *311-41. F:Infinax.At,veCEG'),,.AtCpt+pv.vCt+,Pv Dem. F.*311'25.*310-16.)F: Hp.). 0nv"A C (I Cnvllt) +,(ICnv,"v). [*311'33.*310-17] D.aC + (1) Similarly F: Hp. D. vCu -ipv (2) F.(I). (2).) F. Prop *311-42.F:nfax.LvG(..A+ eC9 Demn. F.*311-27.*310-16.)F:Hp.). (I Cnv"AtI) +, (ICnvllv) eC. [*311-33.*310-16] D./.t +pv e C'"E:), F. Prop *311-43. F:pe C'60).A+Pt'O1=0 t Dem. F. *311-11.:)F: Hp.:).At-

Ipt'09= It[(am). Mept. W=M-i-002} [*308-51] =p: D F..Prop *311-44. F:Infin ax. concord (Atk v).). ut v eC'(Om [*.311P27A4243] *311-45. F:.Infinax.concord(At, v):,u~ffoqov.v=ffOq:).,r&CUr+Pi [*311 25-41'43] *311-51. F: Infin ax.~e D 'He-t A. YeC'H. Y +g9"4C C. C'=D'fH Dern. F.*38-13.) F:Hp. Xe~.:).Y+gX e. F.*306 51.)D F:Hp Pe NC ind. X e4. Y+g (v/l xX) e.Y+gf(v +o 1)/1 X8XJE ~ (2) F.(1).(2). JInduct.)DF:Hp-ve NC ind. X e4 Y+g (v/1x,,X) e (3) F. *305-7. *306 52.) F: Hp. X e4%ZeG'iH.)D.-(2v).,v eNC ind. ZH{IY+g,(v/ x, X)J (4) F.- (3).(4). D F: Hp. Z e C'H. D. Z e:)F. Prop 21-2

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324 QUANTITY [PART VI *311-511. F:Infinax.~,ECO. YEG"H.).(a[X).XE:. Y+gXe,-. e~ [*311-51. Transp] *311-52. F:Infinax.~,YqcG'e.D~(pq Dem. [*311-11] D aY.X + e(P9 *311-53. F Jfna.4fE'~.~[*311P52-33] *311-56. F:.I-nfinax. 4CE0'eg.): 4=4:+~Ps.=E.fl=t'Oq [*311-1430'5253] *311-57. F:: Infinax.:): A.v-~ 6g.q=iO [*31156-1] *311-58. F: Infin ax.,ucCE c'e.) =H6", [*304-3.*270,31] *311-6. F: Inifinax.,l.tv.X,YEv-lz.XHY.Meji.:).M+g(Y-,X),Ev Dem. F.*310-11.:)F: Hp.).MHX. [*308-42'72] D.{[M+g(Y-,X)} HY (1) F.-(1).*311-58. DF. Prop *311'61. F: Infinax. po9v. X = L{J(X, Y). X, Ye v-p. XHY. L Y- XI}.. S'/. +g"6X C v [*311P6] *311-62. F:Infinax.,uv.XiEv-Fl.).(a1Y). YEv-,a.XHY Dern. F.*311V58.) F: Hp.). Xe H"6v-H",:)F. Prop *311-621. F: Hp *311P61.XeC'e0 Dem. F.*308-46.:)F:Hp.:).XCH6,v (2) F.*311-62.:)F:Hp.X,Yev-ja.XHY.:).(aZ).Zev-I-. YHZ. [*308-42,72] D. (HjZ). ZEP -~. (Y-, X) H(Z -,X) (3) F. (3).- *37-1.:) F: Hp.:). X C HlflX (4) F. *308&56-42-72.)D F: Hp. X, YE v-P. XHY. LH(Y-, X). D. XH (X +gL). (X +g L) HY. [*310-11. *308&43]).LeX (5) F. (5). *37-1.. F: Hp.:). H6X CX (6) F. (1). (2).(4).(6). D F: Hp.:). X c D'He - tA- t'D'H1. [*310-12]:). X cG'P:) F. Prop

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SECTION A] ADDITION OF CONCORDANT REAL NUMBERS 3 25' *311-63. I-: Infinax.vEC'O.Xev.NeC'H.).(2[L).LHN.X+gLev Dem. F.E*311-58.)F: Hp..).(a Y). Yev. XHY (1) F.*308-42.)F:Hp.Yev.XIIY.Z=Y-sX.ZHN.).ZHN.X+gZev (2) F. *308&42-72. D F: Hp. Ye v. XHY.Z= Y-, X. NH*Z. LHN.). LHN. X +9 Le v (3) F. (3). *311 58.) H: Hp. Ye v. XHY. Z= Y-, X. NH*Z.). (HL). LHN. X +g Le (4) F. (1). (2). (4). DF. Prop *311-631. F:Infinax.qev.Netp.). (HM, X, Y). M e 1.k. X, Ye v -. XH Y. N = M +g (Y -, X) Dem. F. *31158. *30872.) F: Hp. X e P - t. LHN. X +9 Le v. Y= X fgL. M= N-g L.:. Mc ~u.X,Yev p.XHY.N=M`+g(Y-,X) (1) F.(1). *311'63. DF. Prop *311-632. F: Infin ax. ev. Ne v - t.). (3M, W). Me 1p. M+g W, N +g W e v - p. (M+g W) H (N+g W) Dem. F. *306'52. *311V63-58.:)F: Hp. D. (3 W). We CH. N+gWev (1) F.*311511..:Hp. We CGH. (gM). Me p. M +g W NE/ (2) F.*311-58.)F:Hp.MEl.Nev- u. WE'H.).MHN. WeC'U. [*308172] D. (M+g W) H (N+g W) (3) F.(3).*311-58.)F:Hp(3).N+,Wev.).M+gW WE (4) F.(2). (4).) F:Hp. W e C'H. N +g W e v -,a. (HM). Me /.L 4. M+g We v -,/ (5) F.(I).(3). (5). D)F. Prop *311-633. F: Infin ax. p~v. Ne v. D (21M, X, Y). M e p. X, Ye v - p. XHY. N = M+g(Y-, X) Dem. F. *308 61P4 63.) F: Hp. MHN. X = M +g W. Y= N+g W. D.

N= M+g (Y-s X) (1) F. *311-632. *308-72. D F-: Hp. N rr, B p....HM, W,~ X, Y) - Mep.X=M+gW. YN+g W.XHY.MHN.X,Ye v (2) F.(1). (2). D F: Hp.N eN.p. E (aM,X, Y).Me/J.X, Yev.XHY.N=M+g(Y-,X) (3) F.(3). a*311631.)DF. Prop

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326 QUANTITY [PART VI *311-64. F:Hp *311-61.)v +p X Dem. F. *311-633.). vCs' +g"(1 *311-65.F:nfax [*311V52-64] Dem. F *310-11-111. D F:,iA~v.. ~Cnv"pu) 0 (I Cnv'cv)(1 FI-.(1).- *311-65.)D F:: Hp.)D:. pO,,. -:ICnv"p~e CG'0: (aX). X eGO'E. Cnvc"v =Cnvc",u+pX: *311'73. F:Infinax.XEGe~./)v.).(X+pu)e('X +pv) Dem. F. *311P65.) F: Hp.:). (2p).peC'10. v=/p. [*311-121] D). (HP). peG'C". X+ V = (X +p)+~p (1 F.-*311P27.)DF: Hp. D. X +pa, X +p v~eC (2) F. (1).(2). *311-65. D F. Prop *311-731. F:Ifna.X '~.L~ x~~e(~)[*311173] Dem. F. *311 271. Tas) F: p(1 u'e,~) vp (1) Similarly F X A 'n XP=XPV- (4) F. (3). (4). D F. Prop *311-75. F:.Infin ax.concord (X, IA):X+ X+ [*311-74-43]

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*312. ALGEBRAIC ADDITION OF REAL NUMBERS. Summary of *312. In this number we extend the definition of addition so as to apply to real numbers of opposite sign. As in *308, this requires a previous definition of subtraction. We define subtraction as follows: If there is a X such that v +p X = p, then p -p v is X; if there is a X such that p, +pX = v, then -p v is I Cnv"k, i.e. the negative of X; in any other case,, -p =A. The formal definition is: *312-01. f -p = X {(X\): X,, v C'g: v+pXf =,.XeX.V. I,+pX=v. X C[nv"X} Df Hence assuming the axiom of infinity we have v (~O W.n). -. - - -p, v = (?)X) (v +p X = pu) (*312'18), p (O,,) v.. p, -p v = (?) (p +p I Cnv"X = v) (*312-181), XC'eg. D. X-pX =t'Oq (*312-191). The algebraic sum of ju and v is defined as p +p v if p and v are of the same sign, and as p -p I Cnv"vr if, and v are of opposite signs; i.e. we put *312'02. p +a v = (p +p v) u (-pl Cnv"v) Df This definition is justified because either p,+p v or P-plCnvl"v must always be A. Thus we have *312 32. F: concord (p, v).). +a v = F +pv *312'33. F-: ~ concord (/p, r).). p +a v = p -p I Cnv"v The propositions proved are analogous to those of previous numbers, and offer no difficulty. *312-01. -pv = X {(X): X,L, v eCOg: v+pX= L.Xe X.v.p+pX= v.X eCnv'"X} Df *31202. F +a v = (+pV) v (-p Cnv"v) Df

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328 QUANTITY [PART VI 31281: PP k QUATIT CPART V v+p'X=/p.XeX.v.1p +pX=v.XEICnv""X [(*311P01)] *312-11. F:e -concord (~,v).3). utpv= A [*311'1-27'42-43] *312-12. I- Infin ax.vOp. 3. - P = X t(ix). XE ClO. V +p X = 1. XE X} I A (V +P~X Dem. F.()*3121-6. (2 F. (2). *311-74.)D F. Prop *312-13. F1- Infin ax. pkv.). Ak-pV= Xt(HLX).XeG'0.pu+pX=v.XeICnv""XI -Cnv"QC(X) (P~ +P- X = V) [Proof as in *312-12] *312-14. F- Infin ax.vH,,u = (A) (v +PX= ~ [Proof as in

*312-12] *312-15. F:JInfin ax. pO0,v.) Pt-, v = SX {(X) e C"..p +p, X = v. X e ICnv,"X} - I Cnv"(iX) (pt +pX = v) [Proof as in *312-12] *312'16. F-: E Clog. a)., - P p f0 =,Ft [*312-1. *311-43] *312-17. F:/teC'0g.:).t'Oq p/J.~ICnvl"Ft [*312-1. *311-43] *312-18. 1-:Infin ax.*p (0w 0,)u.) F I z- pv(X) (v +pX=/Ft) [*312-12-14] *312-181. F:Jnfin ax.Ft (O w0,,)v. I)F- v =JICnvl"(X) (I +p X= v) -(A) (Ft +P I Cnv"'X = v) [*312-13'15] *312-19. F Infin ax. concord (X, pt). D. (x, +p F) -p X = Ft [*312-18. *311P65-6643] *312-191. F: Infin ax. XeC'09.). XpX ff09, [*311i52-5343] *312-2. F.- Cnv"ll(F -p, v) =Cnv,"Ft.-p Cnvl"v Dem. F *312-1. *310-16D F:Xe ICnv".t -p ICDnv"v.:,a, veGClog (ajx): X 6G'og: Cnvl"v +pX=ICnv"t. Xe X.v. ICnv"p +pX = (Cnv",v. Xe Cnv,"X: [*3 1132]2:F, v eClog:(X XC'09 v +PI Cnv"X1 =F' X eX. v. O p nv"'X= v. X e Cnvl"X: [*312-1.*310-16]: X e I Cnvl"(/F -,, v):. D F. Prop *312-201. F./FtpI Cnvllv - Cnvl"(t Cnv"F/t -pv) [*312-2]

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SECTION A] SECTIN A] ALGEBRAIC ADDITION OF REAL NUMBERS32 329 *312-21. I-. ICnv"c(v -Pa =,t -P, v Dem. (sx.): X eGClog::t+ X = v. TeX. X = Y I Cnv. v. v+p X=bp. Yet1 Cnv",X. X= YfICnv:. [*310O16] =t, vecC'eg:.-(ajx):X GClg: t+pX = v. X eI Cnv"X -v. v X=a: *312-211. I- 1. u -p I Cnvllv =v -p I Cnv"/.k [*312-201 21] *312-22. F:Infinax.v(ewe,,)U.:)..t-PveC'e Dem. F -.*311-65. *312-12. I-F: Hp. vep.).,a-p v eC'e F. *311V66. *312-15. D I- Hp.. Cnv 6(pk -p v) e C'~,. [*310-16].,) v " F.(I).-(2).) F. Prop *312-23. F:Inflnax.pu(e~we,,)V.).1Lt-~P~eCG'([*312-2122.*31 *312-3. F./. +a V=(G P +V) v (,"-P I Cnv 66V) [(*312-02)] *312-31. F:,- 6(a, v eC'g).).ii +av =A [*312-311. *311 *312-32. F: concord (p,tv).).t+a V= /I +p [*312-311. *311 *312-33. F:c, concord (t, v).).p+av =,iat-p IC nv 6 v [*312-3.-*311Y1 *312-34. F:Jtnfin ax.p, v eG"E)0.:). II+a V eG'eg [*312-3233-2223. *311P44] *312-41. F. lJ+a v= vaI (1) (2) 10O16] I-1] 1-15], Dem. F. *312-32. *311 12.) F: concordQ(,av) v).ta= v +a (1) F. *312-3321. F F: - concord (a, v).). +a V' = Cnv"i(f CUVICv - PP) [*312-201] = v - Cnv"k [*312-33] 'k(2) F. (1). (2).:) F. Prop *312-42. F: Infin ax. concord (Xq,,v)) (X+ z p(X+) p- Dem. F.*311-2 7 42 43)F:. Hp. D:concord (X +pu, X+p v,Xqt,,v): Similarly F:. Hp.): lk +~p=X +PV PP=X+ (2) F.(1). (2). *312-1. DF.Prop

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330 QUANTITY [PART VI *312-43. I-: Infin ax. concord (X,AL,, v). v (O v 0, e), u.). Dem. ~~~~~~~~~~~~(X ~,, P) -P v = X +P (L -,P v) F *311-65-66.:) F: Hp.). (Hp). p e C'g. FL +P jO. [*312-12-1319])..(p). p e C'g.(X +pp) -Pv=X+p p. - p v=p) F. Prop *312-44. F:Infin ax.concord (X,,/ v). (~O,)v: Dem. ~~~~~~~~~~~~(X +P FL) -P v = X-P (v -PFL.*3lP65i36)F: Hp.)(Hp).p EC'eg. V/ + p. [*312-42-19]:). (up). p eG'eg. (X +p t) -p v=X -pp.p = v-p t:) F. Prop *312-45. F:Infinax-.colncord(Xqu).).(X+pFL)-,pF=X-iP(FL-PF) Dem. F. *312-19. *311-43.) F: Hp). FLI -p FL = tfOq. [*311-43] -U [*312-19] = (-ip) -pu p:) F. Prop *312-451. F: Infin ax.concord (X,FLv)) (X +p /FL) -pV (X+a 1A) +a ICnv"v = +a

(FL +ia ICnV"v) Dem. F.- *312-43.) F: Hp. v (O9 w 0.)a +L.(+P) - vX +P (k -,, v) [*312-33] X+p(FL +a ICnv""v) [*312-32-1214] X+a (,O+a CnV""V)(1 F. *312-44.:)F: Hp.Le 3).).(pF)- =X (-) [*312-21] = P X I Cnv"(FL -p v) [*312-33-12-14] = +a(-p V) [*312-33] =X+a(p+a I Cnv"v) (2) F.0 *312-45.) F:Hp. p v.) p. v + /. pv [*312-33-32] =X +a (p ~a ICnv~lv) (3) F. (1). (2). (3).*312-32. *311-43.) F. Prop *312-46. F:Infinax-concord(X,pa).:).(X~ap)+av=X+a(P~av) Dem. F. *312,32. *311~65-66-43.)F: Hp. concord (X, p, v (X +aL)+a v=(X +P P) +p v X% +(,a +a'v)=X +P(+P V) (1 F.*312 451.) F Hp.concord (>,p,IuCriv,'v). (X +a P) +a V= +a(p+a V) (2) F.*312-31.:) F: 01E'eg).D(X +aP) +a A. X +a(+a)A (3) F (1).(2).(3) *311-121.)DF Prop

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 *312-461. I-: Infin ax.concord(tv).)(+a P) +avX+a (p +a v) Dem. F. *312-46.)F Hp.) (v +a FL) +aX V +a (At+aX)(1 F.(1). *312-41.)I-Prop *312'47. F:JInfin ax concord (X,P).)(X +a) +a V= X+a(A+a V) Dem. F. *312-461.)I Hp. (A +a X) +a V Atk +a (X+a v). [*312-41] +a (~u) +a V=A+a(X+a' [*.312'41] =(X +a V) + At [*312-46] =X +a (V +auA) [*312-41] X +a (A +av):)F. Prop *312'48. F:Infin ax.).(X+a/t)+av='X +a(At+aP) Dem. F. *312-31. F.*310O12. DF:. X, veG'eg.)concord (X,,a).v. concord (X,,v): [*312-46-47]:(+oat) +a=X +a (At+a V) (2) F.(I).(2).) F. Prop *312-51. F:Xe C'eg.) X+at'Oq=X [*312-32.*311-43] *312-52. F:Infinax. XeCG'eg.:).X+alCnv"X=t'Oq Dem. F.- *312-33. F: Hp) X +a I Cflv"X=X p X [*312-191] =~'Oq::) F. Prop *312-53. F:. Infin ax.X, veU 6)X — tv..10inv9 [*312-4851 '52] *312-54. F:Jnfin ax.XeGClog.).8 {a.o C'eg.X + u= Dem. F.*312-48-51-52.F: Hp.)X +a(Cnvl"X +~a) =A (1) F. *312-34.)F: Hp.). ICnv"X +,, IeG'e0 (2) F.(I).(2).DF. Prop *312-55. F:.Infin ax.XAt) veG'og.:X +a AL=X +av. At= Dem. F. *312-4153.:) F:. Hp.):X +a t=X+av At(X+a ') +al Cnv""X. [*312A41-48].V +a (+a ICnv",X). [*312-51 52].At=,v:. DF.Prop

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 *312-56. F:.Infinax.concord(Xqh).:):XE~gp~.=.(Hr.aEG'-x+aUlcr, Dem. I- *311P65. *312'32.) F.*311P66. *310-16.) F-:. HP. 2.kE G'0j.:) xg,. (3/IU +PI Cnvlll- = X. [*3125.13332].(HO-).Ue 6C'O A+a 0-= (2) F.*3125351 ~ ~ ~~~~~~~~~~~~~ ~(4) F.(1).(2).(3).(4>)I. F.Prop *312-57. I: Infin ax.,,e llg'e.concord (X, u).:): XghE.(acT). U - C' c'e X +a = Dem. F. *312-4851P52.)-: E'~~C())~Xa~fVXa4 1 F.*3l2-32.*311P27.:)HHp(l).:).(ICnv,"X+apa)EC'e (2) F.() - (2). F: X eGC'n-, eC'O.).-(aO-).u-e6C'O AX+a Tt(3) F- *312-32. *311V27. *310-13.) F (3). (4).:)~ ~ ~ ~ ~ ~ ~ ~~~~~~(4 I:-: Hp.): X eC'On 1- G'3 C. (HO-). UE0 C'OJ. X +aU&.F. Prop *312-58. F-:. Infin ax A, Xq e C'eg.) xo~ a-)-a C' a [*312-56-57]

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*313. MULTIPLICATION OF REAL NUMBERS. Summary of *313. Multiplication of real numbers is simpler than addition, because it is not necessary to distinguish between factors of the same sign and factors of opposite signs. Thus we put *313-01., xa v = X {u, v e C'Og. X e s'. Xg"v} Df Thus if,A, v are real numbers, their product is the class of products (in the sense of *309) of members of,u and members of v; otherwise their product is A. The propositions of this number are analogous to those of previous numbers, and the proofs are as a rule analogous to those of *311, except in the case of the distributive law (*313'55). *313-01. / x v = X tu, ve C'g. X e s'L Xg"v} Df Proofs in this number are mostly analogous to those for addition, and are therefore often omitted. *313-11. F: ~v (A, v e CO9). D. p xa v = A *313'12. F::,, v e CG'. D. Xa v = s' X g" *313-21. -: A, v e C'E v LLt'Oq.. Xa v = s'/A X8"V *313-22. F:,u, v e C'O, v t"Oq.. X, v = s'(Cnv") x, (Cnv"v)) *313-23. F: e C'O,. v e C'.. 1. xa v = | Cnv"s'(I Cnv",u) x/'v *313-24. F: f e C'O. v e,.). p x v = Cnv"s'(p x,)"| Cnv"v *313-25. F. fL xa v = | Cnv"(I Cnv",u x) Cnv) a | Cnv"a Cnv" *313-26. F. p xa Cnv"Y = i Cnv"u xa v = | Cnv"(u xa v) *313 31. F: Infin ax. E '. X e C'H. D. X xg": C H"X xg"t *313-32. F: Infin ax. & e C'. X e C'H. D. X xg": = HI"X xg": *313-33. F: Infin ax. e C'O. X e C'H.. X xg"t e C' *313-34. F: Infin ax. e CO,. X e s Hn,. D. X xg' e C'O

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 *313-35. F: Infin ax. -~e CIO. XeGC'H,,.:).X xg"rr 1 IEG'n *313-351. F:JInfin ax. 4eG'IR,.XeGIH. D. Xxg" C'@n" *313-36. I-: ~E C'eg. a). 0q Xg" C (09oo *313-37. F:X E G'Hg.)D. X x9,"t'O9 =~'Oq *313-38. F:Infi nax. sGeg.10MX eG'Hg. D.X xg"4 eG'E~g *313-41. F: Infin ax concord (/L,v)./A t'Oq.-V +tt'q. D. /XaV EG 'e *313-42. F: Infi n ax. '~concord (/k v). 1) Pe C'ega D.).,LXa VEG'e"n *313-43. F:. p=t,'q0a.v. v = tOq:P, vEG'CIOg::).AUXa V= t'Oq *313-44. F Infin ax. p,PEv'eg.09-).IJXa V 6C'eg *313-45. FP/AXa V=PVXa P *313-46. F:JInfin ax.D.(X Xa P) Xa v=Xa(P Xa v) The following propositions are concerned with the proof of the distributive law. *313-51. F:Infin ax.concord (X, F',).).(v Xa X) +a (V Xa A)= M [(ax, Y, ZZ'). X EX Ye/P. ZZ' V. M= (Z XgX)+g(Z' Xg Y)] [*313,12. *31 2,32. *31 11. *313 41] *313-511. F:Infin ax.X, peGC'e-Z, Z'et. ZHZ' aXE6X a).aZ x9 Z'x XE X X Dern. F.*304-1401.*305-14.:)F:Hp.)D.(Z xgX) H(Z' xgX). [*309-41] D. (Zx9 Z' Xg X) HX. [*311P58])Z xgZ'xgX ex::) F. Prop *313-52. F:Infinax.concord(Xqp,v).:). (Vi Xa X) +a(P Xaii) = Xa (X +aI Dem. F.*313-51511.) F:Hp.)D (V Xa X)+a(V Xa)=M[(ax,,Z).XEX. YepA. ZEV. M(Z XgX) +g(Z XgY)] [*309-37] =M [([x, YZ). XE6X. Y E/. ZEV. M Z Xg(X +9Y)] [*313-12. *312-32.*311-11] = vxa,,(X +a p): D F. Prop *313-53. F:Infin ax.concord (X,pF). '*'concord (X, v). EC'eg.:). (v X X) +a (V Xa tVXa (X+aJ) Demn. F. *313-25.:) F. (X +a Pt) Xa I Cnv"j(X +aOA) Xa ICnv""v} 1 F. *313-52a) F:Hp.) (+a Pt) Xa Cnv 'vP= (X Xa ICnv,"v) +a(/F' Xa ICnv"4v) [*313'26.*311P31] = Cnv"{~(X Xa V)+a (P'XaV)} (2) F.(1).(2).) F.Prop

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SECTION A] MULTIPLICATION OF REAL NUMBERS ~335 *313-54. F:Infiti ax. concord (X, v).concord (X,t) pe E,0 G'g). p Xa (X +at) (V Xa X) +a (V Xa/) Dem. F. *,312-33-34.:)F:.Hp.X+a,/k=p.:):concord(X,p).v.concord(pU,p) (1) F.*313-52.:)F: Hp (1).concord (X, p).). (p Xa V) ~a (I Cnv"/.kxav) =(p +,,jICnvl",s) xaP, [*312-53] =X Xa V g [*312-53.*313,26]:). p x,, v = (X x,, v) +, (a X. v) Similarly F:Hp(l). concord(,a,p).:) p,,=Xv),(xv F.(1). (2). (3).) F. Prop (2) (3) *313-55. F: Infin ax.:). (v xaX) +a,(v xa/JaV =vxa,(X +a/h1) [*313-5250354-11. *312-31]

*314. REAL NUMBERS AS RELATIONS. Summary of *314. In this number we take up the definition of real numbers suggested in *310, namely s"C'eg instead of C'eg. The series of real numbers is now s;@g instead of eg. Everything in this number depends upon *310 32. F:. A, v e C'eg.): s' = S'v.., = v In consequence of this proposition, s r C'g9 is a correlation of the two sorts of real numbers, and the properties of the relational sort can be immediately deduced from the propositions of previous numbers. We define addition and multiplication of relational real numbers so as to secure that, if,p, v are real numbers of our previous sort, the arithmetical sum of h', and h'v is S'(Fu +a v) and their product is s'(p xa v). This is effected by putting *314-01. X +r Y= RS [(3[, v). X = s'*. Y= s~'. R {s'(/ +a v)} S] Df with a similar definition for X Xr Y. The zero of real numbers is now 0q instead of t'Oq, and the negative of a real number X is X JCnv. The fundamental propositions are *314-13. 1: p, V ~ Cog. ~. S'I +r S'v = s (P +a V) *314'14. 1: p, V ~ C' g. ~. se' Xr S~V = S(p Xa v) in virtue of which the arithmetical properties of relational real numbers follow at once from those of real numbers as segments. Relational real numbers are useful in applying measurement by means of real numbers to vector-families, since it is convenient to have real numbers of the same type as ratios. For some purposes, a somewhat different definition of real numbers as relations is more convenient. Instead of deriving our relations from Og, we may derive them from;'Hg, i.e. we may consider the relations si"'C'(Hg instead of the relations s"C1'g. In virtue of *217-43, (s'Hg) ~ (-t'A-t'C'Hg) is ordinally similar to Og; hence the requisite properties of s"C's'Hg follow at once.

SECTION A] SECTION A]REAL NUMBERS AS RELATIONS 337 *314-03. c? = (H?t) E (C'fln -)r [D'(H,) - t~A- tl'C'Hn vi (C'cHn) 4, (tIAOq) Ij (C'Hn vj) r (D'IhE -I - t~P'H) Df *314-04. M+N S~v.M=~f~N=~'.R{ Cta)]Df *314-05. RS v).AJf= ~' N=sc','v.R th?'(,uxav) S`] Df *314-1.. F: A! X~+ Y.:).X, YEh"C'eg [*312'31. (*314-01)] *314-11. F:. Jufin ax.): ft! X +,. Y. =.X, YE h",C',og [*314-1. *312,34] *314-13. F:p e09 el+ ' ', aV Dem. 311.(a314,0). F: Rc'eg. ',=' hlivi(1) F(1).-*310-32.:)F. Prop *314-14. *314-2. F:R el"'(C'O - ~t'09o).). f! R ~Rel num [*310-31] *314-21. F:.Infin ax.):R, SEhl~C,,0g9=. R +rS'E "C'eg..RXr SCEh"C'0! Dern. F. *314-13-14. *312-34.*313-44) F:HIp. R, Se h"C' g.). R +r S, RXr S e "Creg (1) F.(1).

*314-11-12.) F. Prop *314-22. F:RehW'eCg.).R+rOq=R.RXrOq=Oq Dem. F. *314,13 14.) F: pE C'eg. a). i't+r0q = ih'Q~I +a ~'cOq). h'/t" X r Oq = '(It Xc, t'"Oq). [*312-51.*313 43]). c +r Oq = 'P. ~'/A" Xr0q = Oq:) F. Prop *314-23. F: Infinax-Re.c"C'~9.).R+rRICflv=q Dem. F. *314-13. F: uc EC'e0. h. +r ~c CIIVCit = h'(It % Cn v"c). [*43-421]). it +r (O'pl) ICnv = 9'Qk +a Cnv"Pt) [*312-52] - =Oq::) F. Prop R.& W. Ill. i22

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338 338 ~~~~~QUANTITY [ATV [PART VI *314-24. F. R+rS=S+rR [*312-41. (*314-01)] *314-25. F.R XrS=SXr-R [*313-45. (*314'02)] *314-26. F:Infin ax.)(R +rS) +-.T =1?+r (S +rT) Dem. F. *314-13.) F:Hp. p O*, TE6C'eg.aR =S"p.SS= 'oT. T 9'T.) [*312-48] h't p ~, (ar +a.'r)} [*314-13] R +r (S +rTF. *314-11-21. (R +I. 5) +I. T = A - R +r (S +r T)A F (1).(2).) F.Prop *314-27. F:Jnfinax.).(R XrS) XrT=R Xr(SXrT) [*314-14. *313-46. *314-12-21] *314-28. F: Infin ax.)(R Xr S) +r(R Xr T) =R Xr (S r T) (1) (2) Dem. F *314-13-14.:) F: Hp.p, a-reCeg. R=h'p.- S=i'or- T=k'T.) (B xr 8) +r (B xr T) = 6(p x,, o-) +r k~(p xa, r) [*314-21 13] k't(p Xa O-) +a, (p >Xa T)I [*313-55] =h'tP Xa (o+a7-)} [*314-21-14] go,A"0 a [Hp] R xr(S +rT) (1) F. *314-21-11'12. F.(Hjp, a-, -i). /2, 0-, T E (X'ega R = A'p. S = s'cr. T= s'r.) (R xrS)+r (RXrT).R Xr (S+r T)A (2) F.(1). (2).) F. Prop *314-4. F: Infin ax.) 0~' E {(s'Hg) (IAA -L'C'Hg)} mbsmaor eg [*217-43. *304-31282-23. *307-4144-46-25. (*310,01 -01102-03)] *314-41. F. ~ (C'~s'Hg) E 1 -+ 1 [Thie proof is analogous to that of *310-32] *314-42. F: Infin ax.:). h;0,;egsmor eg [*314-441] *314-5. F:.Infin ax.) f! M +, N. M,N M e9"D t H [*312-34. *313-44. *314-42. (*314-04-05)] *314-51. F:JInfin ax.p, v eC'eg9.) h~cf'i t +o, h'd'v = hV'J'Qb +a v).oo 6ic'~ x "~v = h.o7'Gu Xa v) [*314-42.(*314-04-05)] The properties of M +, N and M x0, N result from this proposition exactly as those of X.+,~ Y and X Xr Y result from *314-13-14.

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SECTION B. VECTOR-FAMILIES. Summary of Section B. The present Section is concerned with the theory of magnitude, so far as this can be developed without measurement. Measurement-i.e. the application of ratios and real numbers to magnitudes-will be dealt with in Section C; for the present, we shall confine ourselves to those properties of magnitude which are presupposed in measurement. But throughout this Section, measurement is the goal: the hypotheses introduced and the propositions proved will be such as are relevant to the possibility of measurement. We conceive a magnitude as a vector, i.e. as an operation, i.e. as a descriptive function in the sense of *30. Thus for example, we shall so define our terms that 1 gramme would not be a magnitude, but the difference between 2 grammes and 1 gramme would be a magnitude, i.e. the relation " +1 gramme" would be a magnitude. On the other hand a centimetre and a second will both be magnitudes according to our definition, because distances in space and time are vectors. It will be remembered that we defined ratios as relations between relations; hence if ratios are to hold between magnitudes, magnitudes must be taken as relations. We demand of a vector (1) that it shall be a one-one relation, (2) that it shall be capable of indefinite repetition, i.e. that if the vector takes us from a to b, there shall always be a point c such that the vector takes us from b to c. If R is the vector, the point to which it takes us from a is R'a; thus the above requisite is expressed by "E! R'a... E! R'R'a," i.e. by " D'R C (I'R." It will be observed that the points which are starting-points of the vector form the class (I'R, i.e. the class of possible arguments to R considered as a descriptive function, while the points which are the endpoints of the vector form the class D'R, i.e. the class of values of R considered as a descriptive function. Since D'R C (a'R, we have a'R = C'R; thus the field of the vector consists of all points from which the vector can start. By

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 assuming D'R C ('R, we exclude magnitudes of kinds which have a definite maximum, unless they are circular, like the angles at a point or the distances on an elliptic straight line; but, except when they are circular, such magnitudes are of little importance. According to what has just been said, if R is a vector whose field is a, we have Re 1 -- 1. (IR = a.])'R C a. A relation which fulfils this hypothesis is called a "correspondence" of a, because it makes a part of a correspond with a. The class of correspondences of a we denote by" cr'a," which is the cardinal correlative of "cror'P," defined in *208. Thus we put 4- v cr'a= (1 - 1) 'an D"Cla Df. We proceed next to define a " vector-family of a." This we define as an existent sub-class of cr'a such that, if R and S are any two members of it, R I S= S R. We define a class of relations as " Abelian " when the relative product of any two members of the class is commutative, i.e. we put Abel=Kc(R,SeK. D)Rs.RIS=SIR) Df. Thus a vector-family of a is an existent Abelian sub-class of cr'a, ie. writing "fm'a " for " vector-family of a," we put fm'a = Abel n C1 ex'cr'a Df. The class of vector-families is then defined as everything which is a vectorfamily of some a, i.e. we put FM=s'D'fm Df. Thus a vector-family is an existent Abelian class of one-one relations which all have the same converse domain, and all have their domains contained in this common converse domain. If K is a vector-family, the common converse domain is Lt'["K, which is identical with s'"'Kc, and will be called the " field" of the family. Thus we have F: K FM. -. e Abel. C 1 e 1. s'D"K C s'(I"K. A vector-family may be regarded as a kind of magnitude. In order to render measurement possible, we require various hypotheses as to the nature of the family. Measurement within a given family K is obtained by limiting the fields of ratios to K, i.e. by considering X Kw where X is a ratio, or Z C K where Z is a relational real number of the kind defined in *314. In order to make measurement possible, we wish K to be such that, if X is a ratio, X C K shall be one-one; again, if R, S, T are members of K, and R has the ratio X to S, while S has the ratio Y to T, we wish R to have the ratio X x8 Y to T, i.e. we wish to have X Kr YK:C(XX8Y) K;

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SECTION B] VECTOR-FAMILIES 341 again, if R has the ratio X to T, and S has the ratio Y to T, we wish R S (which represents the "sum " of R and S) to have the ratio X +8 Y to T, i.e. we wish to have (X,'T) I (Y: irT) C (X +. Y), 'T. The above and other similar properties will be proved, with suitable hypotheses, in Section C; for the present, we shall proceed with the theory of vector-families without explicit regard to measurement. The first and most important hypothesis as to a family which we consider is the hypothesis that it is "connected," i.e. that there is at least one member of its field from which we can reach any member of the field by a vector belonging to the family or by the converse of a vector belonging to the family. Such a member of the field of K we shall call a "connected point " of fc; the class of such points will be denoted by " conx'"; the definition is -~ 4 — conx n a = s'('c" n a^ (Sc''a v 'K'a = s'(I"K) Df. It will be observed that s'c'a are the points to which there is a vector from a, while s'Ec'a are the points from which there is a vector to a. The definition states that these two classes together make up the whole field of the family. We define a connected family as one which has at least one connected point, i.e. we put FMconx = FM r K (a! conx'c) Df. The properties of connected families are many and important. Among these may be mentioned the following: If ic is a connected family, the logical product of any two different members of K is null, i.e. if P, Q e. P $ Q, then P A Q= A, or, what comes to the same thing, if P, Q e c, and if we ever have P'x = Q'r, then P = Q; if P e Kc, all the powers of P are either members of K or the converses of members; if P, Qe c, then P I Q is either a member of K or the converse of a member. A connected family may not form a group, i.e. we do not necessarily have P, Qe.:)p, Q. PI Qe, but we shall show at a later stage (*354) that a group can be derived from a connected family K by merely adding to it the converses of those members of K (if any) whose domains are equal to their converse domains. The result of this addition is to give us a connected family which is a group. Another important property of a connected family K is that I r s'(K"c is always a member of it. I r s'^'cK is the zero vector. In a connected family, every vector except I s8'PKc is contained in diversity. For many purposes, the class of vectors excluding I r s8'(I" is important. We therefore put a=K- Rl'I D~

342 QUANTITY [PART VI In the study of a vector-family K, an important derived class of relations is the class of all relations of the form R S, where R, Se K. The operation R S consists of an S-step forward, followed by an R-step backward; that is to say, if R'S'a exists, it is obtained by moving a distance S forward from a to S'a, and then a distance R backward from S'a to R'S'a. The class of such relations as R S, where R, Se K, we call K,; i.e. we put K, = s'(Cnv"K) "K Df. The class K, will have different properties according to the nature of K. We may distinguish three cases: (1) The field of K may have a first term, i.e. there may be a member of s'aI"f which is not a member of s'D"Ka. This case is illustrated, e.g. by a family of distances from left to right on the portion of a given line not lying to the left of a given point. This given point will then belong to s'I"Kc, since there are vectors which start from it, but it will not belong to s'D"ca, since there are no vectors

which end at it except the zero vector. A connected point a which belongs to s (Ic"K but not to s'D"Ka is called the "initial" point, and a family which has an initial point is called an "initial" family. A family cannot have more than one initial point. Thus we put init'c = t'(conx'c- s'D"Ka) Df, FM init = FM n ('init Df. (2) It may happen that, even if K is not an initial family, none of the converses of members of Ka are members of K. (If K is an initial family, this must happen.) This case is illustrated by the case of all distances towards the right on a straight line. It is also illustrated by the family of vectors of the form (+, X) C'H, where X e C'H'. In this case, as in (1), it is possible, by adding suitable hypotheses, to secure that s'Ka shall be a series. This case divides into two, which are illustrated by the above two instances: it may happen, as in our first instance, that the domain of a vector is always equal to its converse domain, i.e. D"K = (C"K; or it may happen, as in our second instance, that the domain is only part of the converse domain. (The domain of (+8 X) G'H consists of all ratios greater than X.) (3) It may happen that ca contains pairs of vectors which are each other's converses. In this case, it is obvious that S'Ka cannot be serial, since R, RBe Ka). R I R I | S'P'K. R I R (I'Ka)2, so that ('Ka)2 is not contained in diversity (except in the trivial case K = t'A). In considering K,, we do not at first explicitly introduce any of the above possibilities, but it is necessary to bear them in mind in order to realize the

SECTION B] VECTOR-FAMILIES 343 purpose of the propositions proved concerning K,. If L is a member of c,, and L = R S, where R, Se K, then if a is a connected point, and L'a exists, it follows that there is a member T of K v Cnv"K such that L'a = T'a. It is easy to deduce from this that L = T, hence L e K v Cnv"/ c. The same holds v v if L'a exists. Hence if E! L'a. v. E! L'a, i.e. if a e C'L, L is a member of K Cnv"K. Thus if a belongs to the field of every member of c,, we shall have c, =K v Cnv"K. We say that a family "has connexity" (not to be confounded with "being connected ") if a! conx'K A P'C"K,; thus we put FMconnex = FM n K (a! conx'K n p'C",) Df, and by what has just been said we have a: K e FMconnex. D., = ic v Cnv"K. We also have F-: Kc FMconnex. D. sK e connex and F:. K e FM conx.: K e FM connex. -. sKa e connex. It is these propositions that justify the notation "FM connex." It is obvious that we shall have g! p'C"K, if D"c = (CI"c, unless K = L'A. Some illustrations will serve to make clearer the nature of the hypothesis a! p'C"',. This hypothesis states that there is at least one term a in the field of K such that, if R, S are any two members of K, we can either take an R-step forward from a, followed by an S-step backward, or we can take an S-step forward followed by an R-step backward. Suppose, for example, that our family consists of all vectors of the form (+, a) NC induct, where A e NC induct. Then if R is the operation of adding pA, and Sis the operation vJ of adding v, R IS is the operation of adding v -c if v >,a, and is the operation of subtracting / -c v if p > v. In the former case R I S e c, while in the latter case S I R e K. In the former case, if r is any inductive cardinal, (R I S)'w = v-c P +c; in the latter case, (S I R)'w = I-o v +c r. Thus in either case e C((R S). Thus the family in question has connexity, and K, = iK V Cnv /c. But now consider the family consisting of all vectors of the form (x pA): (NC induct - 'O0), where p e NC induct - t'O. This is an initial family, its initial point being 1. But it does not have connexity. If R= (Xc,) t (NC induct - '0) and S =(x<, v) (NC induct - A), R I S is the operation of multiplying by v and dividing by A, with its field confined to inductive cardinals other than 0. If v is prime to Ap, this relation has only multiples of At in its converse domain and only multiples of v in its domain. Hence its field consists of multiples of p together with multiples of v. Thus no member of c, except I CS'(I"K, i.e. (x. 1) (NCinduct- 'O), has the whole of s'G(I" for its field, and there is no number which belongs to the

344 QUANTITY [PART VI field of every member of K,. The above family may be usefully borne in mind in considering c,, since it affords good illustrations of most of the general theorems concerning Ic,. If I is any family except tlA, any finite number of members of K, have an existent relative product, and their converse domains have an existent logical product. If K is a connected family, any two members L, M of KC whose logical product exists, i.e. for which (gy). L'y = M'y, are identical, and if x,y are any two members of s'["rc, there is just one member of Kc, such that x= L'y. If Me K, and P is a power of M, there is some member L of I, such that P C L. But P is not in general itself a member of K,. For the application of ratio, the member of K, which contains P is important. We call it the "representative" of P. The general definition of a representative is rep,'P = '(K, n C P) Df. In a connected family, ic, n C P cannot have more than one member; hence if there is any member of K, which contains P, that member is repk'P, and if there is no member of K, which contains P, rep,'P = A. If P I Q is any member of c, (where P, Q e K), we shall have repK'(P I Q)P = PP QP; and if L, M e c,, we shall have rep,'(L | M)P = repk '(LP I M) = rep'(rep'ZLP) I (rep,'MP)}. These two formulae are the most useful in determining representatives. In order to apply the above theory to the measurement of vectors, it is necessary to distinguish between open and cyclic families. An open family is one in which, if Me K, -R1'I, Mpo C J, i.e. one in which no number of repetitions of a non-zero member of K, will bring us back to our startingpoint. If this condition fails, as in the case of angles, or distances on the elliptic straight line, the problem of measurement is more complicated, since, if 0 is a measure of an angle, so is 2v7r + 0 for any integral v. The case of cyclic families will be considered in Section D; for the present, we proceed to consider open families, and we shall still be concerned almost exclusively with open families in Section C. It should be observed that in cyclic families, as we shall define them, members of Ka return into themselves, whereas in open families, not merely no member of Ka, but no member of K,- Rl'I, returns into itself. In most of the families that naturally occur, it happens either that no member of K,- R1'I returns into itself, or that there are members of Ka which do so. But there is no logical necessity in this, as the following instance shows: Consider the family consisting of positive and negative integral multipliers other than - 1, with their fields confined to positive and negative integers other than -1. Then 1 is a

SECTION B] VECTOR-FAMILIES 345 connected point of this family, in fact the initial point. Multiplication by - 1 is a member of K,, since it can be obtained by multiplying by any integer, and then dividing by - p. Also the square of multiplication by -1 is contained in identity, and is the zero vector of our family. Hence there is a member of K, - R'I whose square is contained in identity, although no power of any member of ca is contained in identity. In order to avoid brackets, we put Ka = (K)a Df, i.e. a = I - R1'I. Then the definition of open families is FMap = FM n K (s'Pot"c,a C Rl'J) Df. Hence F:. e FM ap. -: K e FM: M e Ka. M - Mpo C J It will be observed that if K is an open family, K, is contained in Rel num id (cf. *300), and c,a C Rel num. Hence if M e c,a, Mv = M, (cf. *121), and the propositions on intervals in *121 become available. Also if M K,a, and a e s'c(K, we have M M'a Prog. Mpo M*a e. The chief use of these facts is to show that the existence of open families implies the axiom of infinity and the existence of 0,. Hence as applied to open families, the theory of ratio undergoes the very great simplification which results from the axiom of infinity. If K is open and connected, and L, M e c,, and a is any inductive cardinal other than 0, we shall have L=M if L =M- or rep,'L" = rept'M or 3 [! La A My. If p, Tare also inductive cardinals other than 0, we shall have rep,'L = rep,'M7 if LPXCT = MAXC, or if rep,'LPX T= rep,'MaxCT, or if 3!LpXcT A M'xc7. We have in fact rep,'L = rep,'Ma. -. =! LP A Ma.! LPXCT A MAXcT and rep'MP = rep'M. =. MP = My. -. p = a. On applying the definition of ratio (*303-01), we see from the above propositions that, with the above hypothesis, M (p/a) N. =-.! Ma n NP.. repK'M' = rep/'NP, while if R, T are members of K, R (p/r) S.- R. = SP. Further, we have, in virtue of the above propositions, [! La A MP.!, Lr M ".... X a- v X. p, whence X, Ye C'H'.!X,a A Y Kca.). X = Y.

346 QUANTITY [PART VI These propositions, together with X e C'H.. X c, e 1 -1, belong to Section C. They are mentioned here as showing why the propositions of this Section are useful in connection with measurement. We next proceed to consider serial families, which are those in which S'cK is an existent serial relation. For this purpose we require the definition of "FM connex" already given, and also the definition of "transitive" families. We define a as a "transitive point" of K if (s'Ka)"'sca'a C s'Ka'a, i.e. if any point which can be reached from a in two non-null steps can also be reached in one non-null step. We define a family as transitive when it has at least one transitive point. If K e FM conx, the hypothesis that K is transitive is equivalent to the hypothesis that Ka forms a group, and implies that K forms a group. We define a serial family as one which is transitive and has connexity, i.e. we put FM sr = FM trs n FM connex Df. Then if K e FM sr, s'Ka is a serial relation, so that the points of the field of K are arranged in a series by means of relations of distance. When a family is serial, the vectors also can be arranged in a series, by means of a relation which may be regarded as that of greater and less. After a short number on initial families (explained above), we

proceed to the consideration of greater and less (as it may be called) among vectors. We may call a point y " earlier " than a point z when there is a non-null vector which goes from y to z, i.e. when z (sh'a) y. If M, N e K,, we then say that N is "less" than M if the N-step from some point x takes us to an earlier point than the M-step. Writing V,, for "greater than " among members of /,, our definition is V, = MN {M, NE C,: (gx). (M'x) (Saa) (N'x)} Df. For the same relation confined to members of K, we use the notation UK; thus U,= VK Df. If K e FM conx, we have UK = PQ {P, Q e K: (T). T e Kca. P = T Q}; this is generally the most serviceable formula for UK. If K is a serial family, UK and VK are series; and if K is an initial family, U, is similar to S'Ka. The last number in this Section is concerned with the axiom of Archimedes and with the existence of sub-multiples of vectors. The axiom of Archimedes will be expressed by saying that if a is any member of the

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 field of /, and R is any vector, then Rv^a, for a sufficiently great finite v, will be later than any assigned member of the field of K. In other words, putting P= Cnv'S'Ka, we wish to have x e C0P.). (Tv). v e NC ind - 'O. xP (Rv'a), or, what comes to the same thing, P"R*'a = C'P. This will hold if K is a serial family and P is semi-Dedekindian (cf. *214). If, further, P is compact (i.e. 2 = P), then all finite sub-multiples of a given vector exist, i.e. Se. v eNC ind - L'O. D.(L).L ec.S=LV. It will be observed that, according to our definition of ratio, if S=Lv and S: A, L has to S the ratio l/v, so that L is the vth sub-multiple of S. Instead of treating vector-families by the method we have adopted, we might have started from a double descriptive function, which we may denote by x + y, and concerning which we should make various hypotheses. By the general notation of *38, we obtain various relations of the form + y or x +. These relations may replace the Ic employed in our method. For convenience of notation, we may put +'y=+y Df, 4 -+'x==x + Df. Then if + has suitable properties, and 7 is a suitable class, +"-y will be a vector family. Let us assume that x + y exists when, and only when, x and y both belong to the class y, and that when x and y both belong to the class ry, x + y also belongs to this class. Then if x + y exists, so does x + y + y; hence D'+ y C '+ y. Further, by our assumptions, if x, y e, x + y exists, and therefore x e I+ y. Hence ye y.. I'+ y = y. Hence if y exists, D"c+"c e 1. s'D"+"7r C s'(I"+"y. If we now assume x+ y + y x z. x,y,z y = z, then +"^ C 1 - 1. Hence we now have -4 +"y C ex'cr'ry. In order to obtain the Abelian property, we require (x + y)+ z = (x + z) + y, which holds if + obeys the permutative and associative laws. Thus in this case, +" fm'7y.

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 In order that +"y may be a connected family, we require (ga):. z e y. 3: (y)):a = z + y. v. z = a + y. A sufficient, though not a necessary, condition for this is that there should be a zero, i.e. (aa): z e y. 2)z. z

= a + z. In this case, + a is the zero vector, and if a is not the sum of two terms other than itself, a is the initial point of the family. The condition that if x, y are members of y so is x + y secures that +"y is a group. Families which are groups we denote by "FM grp." Thus collecting what has been said, we find that +"y e FM conx grp if + fulfils the following conditions: (1) x +y exists when, and only when, x, y e 7; (2) x,y 7.y. x,y.x+YeY; (3) x+y= x+z.,y,z.y=z; (4) x+y=y+x; (5) (x+y)+z=x+(y+z); (6) (g:a):ze.y~z. z=a+z. From (3) and (4) it follows that the a of (6) is unique, i.e. there cannot be more than one zero. In order to insure that our family shall have connexity, we require further (7) x,yey.Dz,,y:(z):zeTy:x +z=y.v.y+z=x; (8) in order that our family may be an initial family we require that x + y shall only be zero when x and y are zero. With this further condition, our family becomes serial. The above is only a sketch of one of the simplest ways of generating families by means of double descriptive functions. Other ways are possible, and by greater complication greater generality can be obtained. There are some advantages in the above manner of treatment. First, it is possible to take our magnitudes as being the x and y which appear in "x + y," instead of having to take them as the vectors + y or x +. Secondly, our vector-family derives unity from the fact of being generated by the single operation +. Thirdly, the method is more in agreement with current conceptions of quantity than the method we have adopted. The choice

SECTION B] VECTOR FAMILIES 349 between the two methods is a matter of taste; but it would seem that the method we have adopted is capable of somewhat greater generality than the other, and that it requires less new technical apparatus than the other. We have not elsewhere had occasion to treat of double descriptive functions which only exist when their arguments belong to assigned classes, though it is to be observed that our definitions of various kinds of addition and multiplication might quite easily have been so framed as to give this result. For instance, we might have put, +o v-=(xi) t(ac 3)., = Noc'a. v = Noc. r = Nc'(a + 8)} Df. In that case, E!(f +cv) would have implied, v e NoC, whereas with our definition it is only 3! (I ~+ v) that implies A, v e NoC. The general treatment of double functions which only exist in certain cases would require a considerable logical apparatus not required elsewhere in our work, and this is, for us, a reason against adopting the method of treating vector-families which derives them, as in the above sketch, from a single function x + y.

*330. ELEMENTARY PROPERTIES OF VECTOR-FAMILIES. Summary of *330. In this number, we begin by defining the class of " correspondences " of a. A "correspondence" of a is a one-one relation R which makes every member of a correspond to an a, i.e. which is such that, if x e a, R'x always exists and is a member of a. Thus, for example, if p is an inductive number, +,0 L, with its field limited to inductive numbers, is a correspondence of the class of inductive numbers, provided the axiom of infinity holds. (Otherwise, (+o 4) NC induct is not one-one.) The definition of correspondences of a is 4. V *330 01. cr'a = (1 -- 1) n 'la n D"C'a Df I.e. a correspondence of a is a one-one relation whose converse domain is a and whose domain is contained in a. The definition should be compared with the definition of "cror'P" in *208. It will be seen that if Recr'a and x e a, R'x exists and is an a, and therefore 'R'Rx exists and is an a, and so on. Hence all the powers of R exist (*330'23). Similarly if R, S, T,... are any finite number of correspondences of a, R SI T... exists. This is proved for two and three factors in *330'21-22. We define a "vector-family of a" as an existent Abelian class of correspondences of a, where an Abelian class of relations is defined as one such that the relative product of any two of its members is commutative. Thus we put *330'02. Abel =(R, Se. DR,s.R S-SIR) Df *330-03. fm'a =Abel n C1 ex'cr'a Df *330'04. FM= s'D'fm Df It will be remembered that Potid'P and (for certain kinds of relations) finid'P are Abelian classes of relations (*91-34 and *121-352). If P e I -- 1, Potid'P will be a vector-family of C'P, and if further Ppo C J, finid'P will be the same vector-family. One other definition belongs to this number, namely *330'05. K, = s'(Cnv",c) |li Df This definition has been sufficiently discussed in the summary of the present Section.

SECTION B] ELEMENTARY PROPERTIES OF VECTOR-FAMILIES 351 After some preliminary propositions on Cl ex'cr'a (*330-1 —32) and on /C, (*330-4 —43.), we proceed to such properties of families as do not require any further hypothesis as to the nature of the family concerned. These properties are mainly such as assert the existence of relative products, and of logical products of converse domains, or such as assert commutativeness of the relative product under certain circumstances. The earlier propositions deal with members of K, the later propositions mainly with members of /c,. The most useful propositions are: *330-54. F:KEFM.QReKc.E!R'x.).E!vR'Q'x *330-56. F:KEFM.Q,Rec.E!R' a.).R'Q'a=Q'R'a *330-61. F: K EFM - tI/1A. L, M e KIc. 2. a! L n PL 'M. a! DIL I'M. P L n D'M.a D'ILn D'M *i330-611. F: ie e FM - tit"Ai L) M e K, D!.~ L M *330-624. F: Ic e FM-it'iA.Le.)E. A EPot'L *330-63. F:KceFM.L,MCK,.E!Lx.E!L'M'x.).L'M'x=M'L'x *330-642. F: K EPFM-tt 'fA. L, MEK1,. D. (Hx). E! L'x. E! L'M'x *330-71. F:Ke 6FM.P, QcK. p e NC ind - O. E! PP'x. D. E!(P I Q)P'rx *330-72. F:KeFM-t'tfA.L,MEK,.P,a eNCinduct.).a!Us LP rIU'MaC *33073. F: KEFM.P,QEK.peNCind.E!(Pj Q)P'a.).(PI Q) Pf'x-P 'Qp'x *330-01. cr'a - (1 -+) i a1cr n D"ClCa Df *330-02. Abel= (R, SeK.) RS.R IS= S I) Df *330-03. fm'a = Abel A Cl ex'cr'c Df *330-04. FM = s'D'fm Df *330-05. K, = s'(Cnvi"K) "Iit Df *330-1. F: K e Cl ex'cr' a.- =. C 1 - 1. (ICI"K = tf. D"K C Cl'a [(*330-01)] *330-11. F:. (3ca). Ke Clex'cr'a.:c C -+ 1: (aa).P'I = t'a. s'D", C Ca [*330-1] *330-12. F: K eCClex'cr'a. 2.s /= a [*330-1.*53-02] *330-13. F:K EClex'cr'a..D"'KCCClS'P'"K.s'D"K iC s'EU'K [*330P1.12] *330-131. F: (3[a). KIe C1 ex'cr'a. K.# C 1 — * 1.UPKe 1.8'D'%K C S'P"K [*330-11-12] *330-14. F: xE Cl ex'cr'a. 2 D"CK C Nc'a [*330-1]

352 QUANTITY [PART VI *330'15. F-. Cl ex'cr'A = V't'A [*330-1] *330-151. F:a. Cexc'.)A-.EK[*330-14] *330-16. F-: (a1a). K e Cl ex'cr'a Kc~ ifA:. A e[*330'15-151] *330-17. F a! a *K E Cl excr'a.:). D"K111 C Cl ex'"sll'U"K [*330-13-151] *330-18. F:(Ha). K e Cl ex'cr'a: K t'fA::). D"K C Cl ex'ls6('Px [*.330-15-17] *330-19. F- L'(I r a) E Cl ex'cr'a [*330-1] *330-2. F:Ke Cl ex'lcr'a.IREK.aDIM nS'U"K. R M Dem. F.*330-112.) I- lip..! D'M '(FR: F - Prop *330-21. FK e Cl ex'lcr'a. ic +t',. R, S 6K. jRJIS Dem. F.*330-18.)F:Hp.).a!D'IS n s'U K (1) F. (1). *330-2. F. Prop *330-22. F:xeClex'lcr'a.K~t'A.R,STec.:).f!RISIT Dem. F.-*330'21-18.)F: Hp.).f[! D'(S IT) ~sl'U"K~ (1) F. (1). *330-2. D F. Prop *330-23. F: KeCl exlcr'a. ICKtt'A.1ReK.)A Potid,'R Dem. F. *330-18.) F:Hp. P e Potid'fR. f! P.. RI DP n S (""K. [*330-2]). f!R IP (2) F. (1). (2). Induct.) F. Prop *330-3. F: K e Clex'cr'a. IJr etK. K.C S'KI"'K Dem. F. *330-1. D F:. Hp.)D: R EX.)D. R R I Ira:. D F. Prop *330-31. F: KeCl ex'crl'a.-R EK.)R IR Is,'U",c [*330-1] *330'32. F:.KeClex'cr'a.R, SeK.:):RJS=IrS"I'K'.-.-R=S Dem. F.*330-1.)F:Hp.)D.RIRIS=(D'R)JS (2) F.(2).) F:. Hp.): BS=I rS'G('%K.). B= (D'R) iS. [*72-92]) R=Sj r(U'R. [*330-1])R =S (3) F. (1).(3).) F.Prop

SECTION B] ELEMENTARY PROPERTIES OF VECTOR-FAMILIES35 353 *330'4. F: MeK,.=.(aR,S3).R,SCK.M=RiS [(*330-05)] *330-41. F-. Cnv"l,c = c,[*330-4] *330-42. F: Ke CIex'cr'a.I rcea) K. Kc VCnV"' C K, Dem. F.*330o1. *50-5-5l.)F: Hp. REK.).R=(Ir~a)jR.IraE~nV"K (1) F.-(1).*330-441.:)F. Prop *330-43. F K CCl ex'Cr'ca.). I 6(1sOi'cC K, [*330-314] *330-5. F:.KeAbel.=-:R,Se,c.:)RS.RIS=SLR [(*330-02)] *330-51. F K C fM'a. E*K C Abel n Cl ex'6cr'a [(*.330-03)] *330-52. F: K cFMI..(aa).KeAbel nCl ex'cr'a. = K C Abel. K C 1 — * 1. P' K C 1. 8'DII"K C S'Oi "K [*.330-51-131.(*330-04)] *330-53. F:K eFM. Q, ReK.E!R'6Q'x.:).E!Q'Ix.E! R'x Dem. F.*330-5. DF llp.)E!QiR'x(1 F. (1). *30-5.) F. Prop *330-54. F:K eFM. Q, REK. E! R'x. E.E!R'Q' Demn. F. *330'31P5 2 F: Hp.).Rix = R'Q'Q'x(1 F. (1). *330-53.:) F. Prop *330-541. F: K C FM. Q, R C K.). Q""D'R C D'R [*330-54] *330-542. F: K C FM. REC K.:). D"R C sect'8' iK [*330-541. *211 '1] *330-55. F:KCFM-t't'A.Q, Ret.:).2[!D'QnD'R.2[!QcD'fR Dern. F. *330-54.:)F:.Hp.D:xeD'R.D.Q'xeD'R: [*33-43]:):a! D'R.:).a! D'Q et DR (1) F. (1). *330-16.DF: Hp.:).g! D'IQ nD'R (2) F.*330-11-16. DF:Hp.).D'BRCPQ.a!D'IR. [*37-43] D. a! Q"ID'R (3) F. (2). (3). DF. Prop *330-551. F: Hp *330-S55. ft I~ Q IR [*330,55. *37-32] R. & W. iii. 23

354 QUANTITY [PART VI *330-56. J-:KeFM.Q,Relc.E!R'a.:).R'Q'a=Q'R'a Dem. F-. *330-511.:)F-: Hp.:). Q'1?'R'a=R'Q'1?'a. [*Z2-24]).Q'a = R'Q'1?'a. [*330-31-54]).R'Q'a-= Q'R'a~ F-. Prop *330-561. F-KEcFM. Q,RCK.:).JRJQrD'R=QJR [*330-56] *330-562. F-:KEFM.Q,ReK.:).R;QC'Q [*330-561] *330-563. F-:KEFM RE6K.XC

K.:).R~h'X C-'X [*330-562] *330-57. F-:KeAbel.RS, Si. v ENOinduct.:). R"fS=(R 5')".R1S"=S"lR Dem. F -.*301-2.)F. ROIS RISO 0=S (1) F -.*330.5.*301-21.:)F-: Hp. R ISI'=SvIR.:). RSv+cl =Sv,+clIR (2) F -.(3). *301-21.) F-: Hp.:). Rv+01 Sv +cl' = R" IS RI S (4) F-.(4).*301-21.:)F:HpRvISV=(RIS)v.).Rv+clISv+cl=~(R S)v +el (5) F-.(1).(5).Induct.:)F-:Hp.:).RvfISv (RIS)v (6) F-. (3). (6). D F-. Prop *330-6. F-:KE cFM- t't'A.LE K1.)jL Dem. F-.*330-164.) F-:Hp.:). (Q, B). Q, BEI. t! R. ~L=R Q. [*330-54])(5[Q,R, x). Q, R eK.E! R'Q'x. L =RQ. *330-61. F-:KEFM-ftL'A. L,M-K,.:). H! PIL C PM. a! D'L A PVM. -! PIL ADIM. a! D'L A D'M Dem. F-. *330-55-4.)D F-: Hp.)..(Q,R, S, T). Q,BR,S, TE K.L =BRIQ. M= T IS.a I D'BR nD'T. [*330-54])(HQ, BR, S, T, x). Q, BR, S, T E K. L B Q. M =1 T8. E! R(BQx. E! T'S'x. [*34-41]).(ax). E I L'lx. E! M'x. [*33-4:3])!. GIPL A PM (1) F-.(1). *330-41 D)F-.Prop

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SECTION B] ELEMENTARY PROPERTIES OF VECTOR-FAMILIES35 355 *330-611. I-:KcFMf-tL'LA.L,ME/KL.:)f[!LJM [*330-61.*34-3] *330-612. F-:KcFM-'t'tlA.L,MN6 K. G.'L ^ U'M nUGN Dem. F. *330-22-4. D F:lip.).(IP, QR, S, T,W).P, Q,R, S, T, WIei. L=PjQ.M=RjS.NT=TjW.ft!PjRjT. [*330-53]) (.(PQ, R, S,T, W, x). P, Q, R, S, T, WE6K. L =P IQ. M =JRIS. N= T IW-E! P'x.E! R'x.E! T'x. [*330-54]).(ax). E! L'x. E! M'x. E! N'x:) F-. Prop *330-613. F:KeFMV-t't'A.L,MNE K,.).!LIMIN Dem. F. *330-224.)D F: Hp.:). (HP, Q, R, AS, 17, W, x). P, Q, R, AS, T, W c K. L=Pj Q. M=RjAS.N= TJW. E!P'R'T'x. [*330-"5)4]). (HfP, Q, R, AS, x). P, Q, 1?, SEK L= P IQ. M=RI ~S.E! PcRc(Nc'x). [*330-54] D. (HP, Q). P, Q e K. L = P IQ. E! P'(M'N'x). [*330-54] D. (ax). E! L'M'N'x: D F. Prop *330-62. F:KceFM.L6K,.AS6K.:). ASLC-LjS Dem. F. *30'0561.:) F: Hp.P, Q6eK.L = P Q.).AS P C P IS. [*330-5]:) Sj QC P QIS: F. Prop *330-621. F:K cFM - tffA. L e /.C'P C sl'P'K.c P: AS K.s.S IP C P IAS: I j P IL Dem. F. *330-11.)F.Hp. Q, R eK.L =Q 1?.R xPy. D (au, z). uRx.zQy. xPy. [*34'1] D.ftRjPJQ. [*330-561] D.ft!PjQjIR. [Hp] D. ft! P IL:. D F. Prop 23-2

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356 QUANTITY [PART VI *330-622. H: Hp*330621.!.!L I P Dem. H. *33011. *7259.) H Hp. Q, R E K L= Q I R.. P C. QI Q I Q Q [*72-59] D. PIQCQIP. [*330-621] ft!QPI?R [*330-5] [Hp] D. ft! L IP: D. Prop *330-623. FH: CEFM.SeC IC. L eK,.Me Pot'L.).S1i MECMIS Dem. H. *34-34.) H: Hp.SI MCM I S.:). SI MIL C MI SI L. [*33062] D. SIMILCMILIS (1) F.(1).*330-62. Induct.DF. Prop *330-624. H: K EFM- t't'A. LEK4.).AriJE Pot'L Dem. F~. *c330~66. DFH.~t(1) H. *330-622i623. DH:FHp. Me Pot'L. jf! M.).!M IL (2) H. (1). (2). Jnduct. D H:. Hp. D: Me Pot'L. Dm. t! M:. D H. Prop *330-625. F:Ce FM.L,MeK,.QePot'(L M).SeK.).S QC'QIS Dem. F.*330'62.)F:Hp.).S LIMCLj MjS (1) H. *34 34. H: HP. Re Pot'(L 'M). SIR C S. R).SIRILLM C ISIL M [(1)] CRILIMIS (2) F.(1).(2).Induct. DHF.Prop *330-626.: KeFM-t't'A.JL,MeK&,.).Ar-se Pot'(LIM) Dem. F. *330-611.:): Hp..ft!LIM (1) F. *330-621-6225.)F: Hp. Qe Pot'(L I M).!tQ.).!QIL (2) H.*330-625. H::Hp. QePot'(LIM).SeK.).SIQILCQISIL [*33062] C QILJS (3) F.(2).(3).J*330d621cD.)HF: Hp. Qop Pot(L I M). f!.:. t!Q LI (4) F. (1). (4). Induct.:) F Prop

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SECTION B] ELEMENTARY PROPERTIES OF VECTOR-FAMILIES 357 *330-627. F: KeFM-t't'A.L,MeK,.PePOt'M.:)4 jj!PjL.f[!LIP Dem. F. *330-623.)F: Hp. SC K.)SIP IL CPj8jL. [*330-62])SI P IL CPIL IS (2) F. (2).*330-622.)F: Hp. f! PJIL.:). f!MljP IL (3) F.-(1).(5). Induct. DF: Hp.D. f! L IP (6) F.(4).(6). DF. Prop *330-63.:I-:EFM. L,ME6K,.E!L'x.E!L'M'x.:).L'M'x=M'L'x Dern. F. *330'56.:) F: Hp. Q, R, S, TE KC. L =Q I R. M= S IT.:). Q'R'S'T'x = Q'S'1?'Thx [*330-5] =S'Q'T'R'fx [*,330-56.Hp] - S'T'Q'R'x::) F. Prop *330-64. F:.KeFPM. L),Me.K, E! L'x.E! L'M'x. E_ EM'x. E! M'L'x [*330-63] *330-641. F:.KeFM.L,MeK,c.E!L'x.E!M'x.): E! L'M'lx.. E! M'L'lx.. L'M'x =M'L'x [*330-63-64] *330-642. F:KeFM-t'tfA.L,MeK,.:).(2[x).E!L'x.E!L'M'x Dem. F. *330-21.) F: Hp.). (a], Q, R,5,x). P, Q,B, Scc. L= P IQ.M=R IS. E!P'R'x. [*330'53-54]). (HP, Q, R, 5, x). P, Q, 1R, Sc 6. L = P I Q.-M = R IS. E!IP'Q'x. E! IP'Q'(R'Sfx::)F. Prop *330-643. F: Ke FM. Pe K. LE, 1E!L'x.).P'L'lx=L,'P'x [*330-56-5] *330-65. F:KcFM. Q,R, 5,TEKc. R'Q'x= T'S'lx.).T'Q'x= R'S'x Dem. F. *72-24. DF: Hlp.:).Qtx = R'T'S'x [*330-56] = T,'R'S'z [*72-24] ~.T'Q'x = R'S'x: D F. Prop

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358 QUANTITY [PART VI *330-66. F:.xceFM.Q,R,STeKc.E!R'Q'x.E!T'S'lx.2: 1?'Q'x = T'S'lx. =. T'Q'lx = R',S'x Dem. F *330-56. 2) I- Hp. T'Q'x =R'lS'x 2. T'R'Q'x = R'R'S'x [*72-241] = SIX. [*72-241] 2.R'Q'x = T'S'x (1) F-.(1). *330-65. F. Prop *330'7. F: FM.-P, QeK. p eNC ind - 'O.E! QC (pi Q)Pc"P'~x. Dem. ~~~~~~~~~Q'(P I Q)p c1'~x = (P I Q)P'x -. *330-56. *301-2. D F-: Hp. E! Q'(P IQ)0'P~x.2 Q'(P IQ)0'P'x - (P (1)I C F-. *330-56. *301-21.2 F:. Hp: E!I Q'(P IQ)P-OI'P'x. 2" QC'(P IQ)Pcl'"P'x = (P IQ)P'lx: 2: E! Q'(P I Q)PfP'x. 2.Q'(P IQ)CP'Px = (P IQ)P'Q'P'x [*330-56.*301P21] = (P I Q)P+"lx (2) I-.(1). (2). Induct. 2 I-F. Prop *33071. FKF.,ecpNidtOEPf.)E(I)c Dem. F.*330-54. D:Hp.E!Pl~x.2E!(P IQ)"x(1 F-. *301P21. 2 H:. Hp: E! PP'x. 2,. E! (PI Q)P'fx: 2 E! Pp+.1', 2 E!i (P I Q)P'Plx. [*330-52] 2 I E Q'(P IQ)P'P'x. [*330-7] 2. E (P I Q)P+clifx (2) F- (1). (2). Induct. 2 F-. Prop *3301711. F:c e FM. Q e s'Pot"x. 2. =C C(C/ Dem. FL*330-52.21F: Hp. P eK.2D.U'CP = s'U"Kl (1) F. *37-322.2 F:Hp. P e K.Q ePot'P.d'IQ sGx2GI'(Q IP) =sUx (2) F- (1). (2). Induct. 2 1F. Prop

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SECTION B] ELEMENTARY PROPERTIES OF VECTOR-FAMILIES 9 359 *330-72. F-: KEFM-t't'A.L,ME6K,.p,a-eNCinduct.:).3!c'LPerJ1'M"' Dern. I- *330-711P23.:) I- Hp. P, R E Kc ({a). E! Ry'a. Ro,'a E GYIP"I. -.(2). *330-71.) I-: Hp (2). Q, SE K. L = P IQ.M = RB S.).E! LP'tx. E! M~Ix. [*3:343])xeUYILP APGMa (3) F - (1).(3).)I-. Prop We have " NC induct " in the above proposition, not " NC ind," because it is necessary to have E! LP. E! MO', and by *301-16 this may fail if either p or a- is null in the type of L and M. Thie existence of a family does not imply the axiom of infinity, since the family may be cyclic. *330-73. F-:KEFM.P,QeK.peNCind.E!(PIQ) P'x.:). (PjI Q)P'x-Pe'QP'x Dem. F-.*330-56.:)F:Hp.E!P~y.:).Q'P y = P'QY (1) F -. (1) F)I- Hp. QPP —"xc= Pp-o"Q'x. E! PP'Y.) QrPPry =PcQ6PP-Ciy [Hp] = PCPPo1CQC y [*301P23] =PPCQt Y (2) F-. *301g23. 2)1-: Hp. (P IQ)P'x- =PP'QP'x. E! (P IQ)P+ol'x.. (P IQ)P+C1'x = P'Q'PP'QPCx [(3)] = P'PP'QCQP'x [*301-23] - PP+ol'QP+Ol'rx (4) F -. (4). I nduct. 2) F-. Prop

*331. CONNECTED FAMILIES. Summary of *331. A i connected point " of a family K is a point of the field of K from which every member of the field can be reached by a member of K or the converse of a member. That is, if a is a connected point, we are to have x e s'("c.:. (aR). R K. (R;) a as well as a e s'(['c. This amounts to saying that every member of s'AI/c is of the form R'a or R'a, where R e K. The definition is *331*01. conx'K = s'aIK n a(g'c'a v S'c'a = s'(Ic) Df Here we include the factor s'n'cK in the definition, in order to exclude the case when Kc= -A. If s'aI"c were not included, we should have conx'tA = V, whereas with the above definition conx't'A = A. In the case of any other family, the factor s'c"K makes no difference, since if s'(["c exists, sc'a v Sc"a = S='(IK.). a e C'ScK, and if K is a family, CG"K=s' I'K. But in the case of t'A, the factor s'Ci"C insures that no connected point exists, thus securing, conversely, that a family which has a connected point is not tLA. This is convenient, since the case of t'A, which is trivial, would often otherwise have to be explicitly excluded. The definition would be more analogous to the definition of a connected relation in *202 if we put conx' = s'C(K n a (S'Ka'a u 'sca'a v tia = s'a"C) Df. But this definition fails to give us the information that there is a member of K which relates a to itself, whereas our definition does give this information, and hence leads to the proof that I r s'GI"K e K, i.e. that there is a zero vector. We say that a family " is connected" when it has at least one connected point, i.e. we put *331-02. FM conx = FM n C (a! conx'K) Df

SECTION B] CONNECTED FAMILIES 361 When all points of the field are connected points, the family " has connexity" (cf. *334'27), provided Ic t'A. For the present, we only assume that at least one of the points of the field is a connected point. To take an illustration: the family whose members are of the form (x /t) C (NC induct- t'O), where it e NC induct - t0, has only one connected point, namely 1. If we had taken positive and negative integers, both as multipliers and as constituting the field, we should have had two connected points, namely 1 and - 1. Almost all our future propositions on vector-families will be confined to connected families. In the present number, we prove first that in a connected family K, the vector which relates a connected point to itself also relates any other member of the field to itself (*331'2), whence it follows that I r s'K",c is a member of K (*331'22), and that every other member of K is wholly contained in diversity (*331'23), and that K v Cnv"Kc C K, (*331-24). We next prove that the product of two members of Ic is a member of K or of Cnv"/ (*331*33). We then proceed to consider K,, and prove at once the two fundamental properties of K, in a connected family, namely (1) that between any two members of s'(I",c there is a relation which is a member of K, (*331'4), and (2) that two members of K, whose logical product exists are identical (*331'42). From these two propositions it follows that there is just one member of K, which relates any two members of s'aI"Kc (*331'43). Finally we prove that any power of a member of Kc is a member of Kc u Cnv"K/ (*331P54), and that any power of a member of K, is contained in some member of Kc, (331-56). Stated symbolically, the above propositions are as follows: *331-2.:. K e FM. a e conx'K. x e s 'i'c. R e Kc. 3: R'a = a. -. R' = x *331'22. F: e FMconx. D. I rs'(sIce K *331-23. F: h e FMconx. D. K C RI'I v RI'J *331-24.: K e FM conx.). c v Cnv"K C,/ *331'33. F: K e FMconx D. s. sc {K C c C Cnv"K *331'4. F: K e FM conx. x, y e s'"CKc..(aL). L e c,. = L'y *331-42. F:. K e FM conx. L, Me K,. D: L h M.. L = M *331'43. F: I e FM conx. x, y e s'(".. M (M e Kc,. xMy) e 1 *331'54. F: K e FM conx. P e K.. Pot'P C Kc u Cnv" *331'56. F: e FM conx. L e K,. M e Pot'L. D. (AN). N e K,. M N *331-01. conx/' = s',(c r, a (s'a'a u s'Ic'a = s'(I"c) Df *331'02. FM conx = FM, <c (ra! ncn xc Df - -- --- - - -.,,I * %W",. A.

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362 QUANTITY [PART VI *331-1. -: a E conx',K a e s"[" "c'a V s'KI'a = s'P',c [(*331.01)] *331-11. F-:. aeconx'ic- ~aes'dI"K:xES'"KI"C:. (H)R e- (RR [*331-1] *331-12. I- a2! conx'K..,3 ~ Kt'A [*331 1] *331-13. F-: K e Cl ex'cr'a.) a e conX'Kc K ic VI. 8K'a v 9'K'a = A 6I Dem. - ~ 4 — - I -. *53'24.:) F-: Hp. 1ic +$ t'A. a v 91K'a - ",. a! 9'i'a v Ka [*330-13].a es'P'K (1 *331-131. F-:: KEClex'cr'a.):. aeconx'K.=: KtfA:xcs'(I""Kc.D (aR).REi.x(Rv~R)a [*331-13] *331-14. F:.X==KV CnvI",c.):aEconx K.=.aes'OI"K.s'X'a s'P'CK [*331 1] *331-2. F:K cFM.a conxucx'K. Xs'U"KIc. R eK.)R'a = a ER'x= x Dem. F-.*331-11.)I Hp. (HS).Se6K - X(S S) a (1) F -.*330-5.)F-Hp.S eK. = S'a. R'a =a.). R'x =S'IR'a [Hp] = S'a (Hp] = x (2) F. *330-56.)IF:Hp.S et. x = S'a. R'a= a. R'x =S'R'a [Hp] = S'a [Hp] = x (3) F. (1). (2).(3).) F:Hp. D:R'a a.D. R'x = x (4) Similarly F:Hp. D:R'xxa.)R'a =a (5) F-. (4). (5).:) F-. Prop *331-21. F:-:KcFM.aeconx'K.ReK.-):R'a=a.=-.Irs'U["K=R Dem. F-. *331-1.:)F-:Hp-IrS'(1"K==1?.).R'a=a (2) F-. (1). (2). ~F. Prop *331-22. F- Kce FM coflx). Irs'("KIIIceK Dem. F-.*331H1. DF:Hp. a econx'Kc. D(aIR). R e K.R'a =a (1) F-.(1).*331V21.) F-.Prop

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SECTION B] SECTION B] CONNECTED FAMILIES36 363, *331-23. F: xEcFM con x.,C RI'IIvwRI'J Dem. F. *331-221. D F: Hp. R E K. ft! R A I.). R C I:) F. Prop *331-24. F: ice FM conx.). K w CnviCC C ic, [*330'42. *331P22] *331-25. F:/ cEFMCOnx -1.). Hic iRII'J [*331~22'23] *331-26. F: Kc eFMconx -1 I.K &'K, I —L'

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' KE, Dem. F. *331P2225. F F: Hp.. (aa, B, S, x). R, S EX. a~a.- aSxa. a+ x [*331P24]). c,c E. II -p+ 1 (2) F.-(1).(2).-*330-52.) F. Prop *331-31. F:.XEFM. aE conx'C. XE6S'U"X.-PEX.KNEK,.D: Pla =N'a.. P'x =N'x Dem. F*331P11. *330-4.) F Hp.).(HQ,BR,S). Q,R, 8e K. X(QW-Q) a. N B S(1 F *3305.) F: Hp.- Q,R, SEKt. x= Q'a. N=BJIS. P'a=N'a.P'x = Q"S'a [*330-56] = B'Q'S'a [*330-5] = RCS'Q'Ca [Hp] =1IV'x (2) F*330-56.) F: Hp. Q, R, SE K x=Q'a. N =RIS. P~a = N'a. D. P'x = Q'R'S'ca [*330-5] =BRCQ'S'a, [*330O56.Hp] = RCS'Q'a [Hp] = N'x (3) F()(2).(3).) F:Hp.P'a = N'a.).P~x = N' (4) Similarly F: Hp. P~x = N'x.).P'a = Na (5) F.(4). (5). DF. Prop *331-32. F:.XeFMCOnx.PeXe.NEXc,.):f[!PAN.=-.P=N Dem. F. *331-31. D F:: Hp. aI E conxlXc. D:. x;, y e s8'(I"X. D P'x=N'lx.=.P'la=N'la.=.P'ly=N'ly (1) F.(l).(*331P02).)F:. Hp.):x,ycs'P'IIc.P'x=N'x.).P'y-N'y: [*33-45.*72-94]:):fj! PAN.).P=N (2) F.*331-12.*33O16.:)F:.Hp.):P=N.:).f!PAN (3) F. (2).(3).:) F. Prop

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 *331-321. F:. xEFMconx.P, Q cK.):!P AQ..P =Q [*331-32-24] *331-33. F ice.FM conx.).8si "/C "C KuCnV"CDV Dem. i -.*330-5.) F: Hp. P, Q,R 1eic.- P'Q'a =R'a. S E K. y = Sa.). P'Q'y = S'P'Q'a [HP] = S'R'ca [*33O-5. Hp] =c (2) I-.*3305 6.~ I- Hp. P, Q, 1? e K. P'Q'ca = R'a. Se K. = S'a.).P'Q'y = S'P' Q'a [HP] = S'1?'a [*33O-56.Hp] = RCy (3) Similarly F-:Hp. P,Q,1RE K.P'Q'a =R'a.)PjIQ =R (5) I-.Hp. P, Q c.)1 (aR):Re1?EK:PI Q=R.v..P Q =R:.) I F.Prop *331-4. FH i cFM conx.wx, y es',c.""K (aL).L i,c4. x= L'y Dem. F. *330O56.~ DF: Hp. R, SeK tx =1'a. y= S'a.:). x= S'Rcy [*330-4].(HL). LE K,. xLy (2) F.*331-24-33.~ F: Hp. R,Se i. x= R'a. y= S'a.). RIS IcK4. x= (R IS)'y (3) I-.*331V24-33.) F-: Hp. R,S~eKc.=R'a-y=,S'a.:) 1RjS6K,.w=(RjS)'y (4) I-.*330-4. F: Hp.R,S~eK. x=R'a-y=,S'a. D.R1SeK~.x=(RjS`)'y (5) *331-41. F - K e FM conx. 'KC, = (8'P'ic) T (S'P'"K) [*331-4] *331-42. F:.KxeFMconx.LMel,.:):A[!LAM.=E.L=M Dem. F. *330-6.*331-12.)F: Hp. L= M.:). f IL AM (1) F.*331,4.) F Hp. Lx= M'1x. E! L'y. D (jN). N e Kc. N'x = Y. E I L'y.

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 [*330-63]).(3N). NEK,(. N1x = y. L'g= NIL'x [Hp] = N'AI'x [*330-63] = M'N'x [*13-12] L'y = M'y (2) Similarly F:Hp. L'x=M'lx.E!M'y.).L'y=M'y (3) I-. (1). (4).:) 1-. Prop *331-43. F:KEFM conx.x, y 6s'G"Kc.M (M 6 4. XMy)E1I Dem. F.*331-4.) F: Hp.).(aM).(Me,cK.XMy) (1) F.*3 3142.) F:Hp.L, ME K, xMy.xLy..L =M (2) i-.(I). (2).:)F. Prop *331-44. F:. KeF3Iconx.P,QeIc.:):f!PAQ.=.P=Q [*331-42-24].*331-45. F:.KcFMconx.LMN6 K,.:) Dem. ~ ~ ~ ~ ~ 1j!LIMAN.=E.LJM=NVc1'f(LjM) F.*33O-611.)F:Hp.LfM=Nr (G'(LIM).)4f1!LIMAN (1) F.*330-63.) F: Hp.L'M'lx=N'lx.E!L'M'y.XeK4.y=X'x.:). L1M1y = L'X'M'x. E! L'M'x.E! L'X'M'x.E! X'x. [*330-63]).L'M'y = X'L'M'x. E! X'x. [Hp])L'M'y = X'1Nx.E! X'X. [*330-63]).L'M'y = N',X'x [Hp] =NIVy (2) F. (2).-*331-4.)DF: Hp. L'M1x = Nx. y eG'(L IM).). -L'M'y = Ny (3) F. (1). (3).:)F. Prop *331-46. F:.Hp*331-45.):MJL=NrG'I(MIL).=-.LIM=NrG'I(LIM) Dem. F.- *330i642-63.) F: Hp. L IM= N r P1(L I M).:). (ax). M'L'x = N'x. [*331P45] D.

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MlL=Nr~G'(MiL) (1) Similarly F:llp.MIL=Nr'P"(MIL).D.LIM=NrqI'(LIM) (2) F.(I). (2).:)F. Prop *331-47. F:KeFMconx.L,MeK&,.).(aN).NeK,c.LlMC-N.MILC-N [*331-4645-4] *331-48. F: KeFM. LIE Ka! conx'K nC'L ~L e c vCnv"',K Dem. F. *330,41.) F:. Hp. a is conx'c ^ CIL. D: L, L e /,~: E! L'a. v. E I Pa: [*331-11] L): LEK,: (aR): R e v Cnv"x:L'a-=R'1a. v. L'a =Ra: [*331P24-42]):(aR): R E KC v Cnv"Kc: L = R. v. L =:D F. Prop

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366 QUANTITY [PART VI *331-5. F': IeFMCOnx. PE6K. LEK,.:).LIP,PjLeK4, Dem. -. *331-33.) F:Hp.Q,Rexc.L=QIR.:).(a$)./SE~vCnv",lc.LIP=QIS (1) F. *330O4.)F: Hp(l). 8 ex.LjP=QjS.).LJPei,c (2) F. *34-2.) F:Hp (1). eCnv"'K.L P =Q S.). (T). T Eic. LP =Cnv'(T Q). [*331P33]:). L IP e I Cnv"'K. [*331P24]:).LjPetK, (3) F-. (1). (2). (3). *330-41.) F. Prop *331-51. I K E FMconx.PE/K.D). Pot'P C/K, [*3315 ". Induct] *331-52. F:KceFMconx.P,QIeK.Lex4.m).PILIQEK4, [*331P5] *331-53. F: Ke FMconx. P.Qe6K -p, c eNC induct.).PP IQ'E K, [*331-5. Ind uct. *331P51. *330-43] *331-54. F: ice FMconx. P e/.:). Pot'PC K vCInv"Kl/ Dem. F. *330-711 F): Hp. aeconx'K. Qe6POt'P.:). E! Q'a. [*331-11]).(21T). T E K v Cnv"Kc. Q'la =T'a. [*331-51-4224]).Q e Kc v Cnvl"K:c F. Prop *331-55. F:Ke FM conx.P, QEK,. p eNC induct.) (PJI Q)p C PPjIQP. PP IQP e 1c, [*330'73. *331-53] *331-56. F: KeFMconx.LeK1,.MePot'L.:).(a[N).NeK,. MC-N [*331P55. *330-4]

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*332. ON THE REPRESENTATIVE OF A RELATION IN A FAMILY. Summary of *332. We saw at the end of the last number (*331'56) that any power of a member of K, is contained in a member of K,. When a relation is contained in a member of K,, we call this member the "representative" of the relation in the family. For purposes connected with the application of ratio, the "representative" is an important function of a relation, especially when the relation concerned is a power of a member of s,. By the definition of ratio (*303'01), we shall have L (pla) M if a! Lr A MP and p Prm ra. Now if La and MP each have a representative, then they must have the same representative if 3! LK A MP (by *331'42). Hence we are enabled to substitute an equality for X! Lr A MP in dealing with ratios of members of E,. The elementary properties of representatives are dealt with in the present number. We denote the representative of P in the family K by " rep,'P." In order to insure E! repc'P under all circumstances, we do not define rep,'P as the only member of K, which contains P, but as the logical sum of the class of members of K, which contain P, i.e. we put *332-01. rep,'P = s'(Kc, n P) Df In a connected family, if P is not null, K, n C'P cannot have more than one member (*332*21), and therefore the representative of P, if it is not null, must be a member of c, (*332'22). If P is a member of K,, it is its own representative (*332-241). We prove in this number that, if P, Q, R,... have existent representatives, the representative of their relative product (unless this product is null) is the representative of the relative product of their representatives (*332'37). Among other important propositions in this number are the following: *332'32. F: / e FM conx. L, M e,.. rep,'(L M) = rep('(M IL) *332'51.: /C eFMconx. P, QEK. D. rep,'(PIQ)= QIP *332-53. H: K FM conx. P, Q e K. p e NC induct. D. rep,(P I Q)P = PP I QP *332'61.: K e FM conx. L e K,. 2. rep,"Potid'L C K,

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368 QUANTITY [PART VI *332-8. F: KEcFMconx. LM,MEKl.V e NCind.:). rep~'(L - ~ rep '(LP I Mt) *332-81. F:KcFM conx-VI CoENC ind - fO. L6K,.). re p/LvxcT, = repK '(rep~lLv)a*332-01. repK'rP A(K,cr C '-P) Df *332-1. F.repK'P = hi 4'P=- L) LEPCxxy [(*332-01)] (,nC)=^9JH).L,.xy *332-11. 1-F rep,t"P.).P C rep,,'P [*332-1] *332-13. I- rep,,A = /c, [*332-1] *332-14. F P C Q. repK'Q C repK'P [*332-1] *332-15. F. repK6P =Cnv'rep PK6 Dem. F*330-41.FK, nC'CP =CnaV"(K, MC'6P)(1 F(1).-*332-1.F.-Prop *332-16. F- K =t'A. rep,'P =A [*332-1] *332-2. F:.K6FM - t'fA.:(KA P) re P Dem. (,nC") rpi FI-.*330-6 F) I Hp.! (K, el')l!(,AC')-iA [*332-1] C'"). C re cP) (1) FI-.(1). *332-12.)F Proprep *332-21. F: IKcFM corx P P.:) +-, ~nCP~O Dem. K CIT 0 F. *331-42. F:Hp. L, M E K,. P C L. P C M.) L = M:) F Pr-op *332-22. F:. Ke FMconx. ip)rep'PE K,. V. repK'PA Dem. F. *332'21-12 F): Hp.! repK'.) (Kc, r% C'P) el [*332-1]).rep,,'P e K,: F. Prop *332-23. F:.KeFMconx.fj!P.): repK'PIEK,.=-.a!(K,n C'P) Dem. F.*332-22-2)F:Hp. repK'Pr eK,.).(K MC'cP) =A (1) F.*330-6. D F:Hp.rep~PK6 K,. t! rep,'P. [*332-2] V!(,C.'P) (2 F. (1). (2).) F. Prop 2

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SECTION B] ON THIE REPRESENTATIVE OF A RELATION IN A FAMILY 369 *332-231. F:.KceFMconx-l.:):rep/,Pe/c,.= f[!P a!(K, C"'P) Dem. F. *331-26.)F:.Hp. rep,,"PCE K,.) rep,'-P + #K [*332'13])Pt+A(1 F.(l).*332-23)F. Prop *332-232. F:. KeFMconx-I.:):rep,,'PEK,.=.jYP f{jrep/,P [*332-231P2] *332-24. F:.KceFMconx.ft! P.:):Le(c1A'C-'P).=.f'rep,,'P.rep,,'P=L Dem. F. *332-2. D F:Hp. D: Le 1c AGC'P.). t! repK'"P (2) [*13-12])t! repK'P. rep,,'1P =L.)LE K, (3) F.(1).(2). (4).:)F.Prop *332-241. F: K eFM conx P. PKL. P P=rep,'(P Dem. 4 -F.*332-24.) F:.lHp. ft P.:P CK, ri C P. ft!rep,,'P. rep,,1'P =P: [Hp] D repk,'P = P (1) F.*330-6.:)F:Hp.r%.t!-P.:).K=t'A. [*332-13] D. repK'P= (2) F. (1). (2).:) F. Prop *332-242. F:K eFM conx.! P. f! rep,,'P.:). rep,'P = rep,,'rep,,'P Dem. F.*33222. F: Hp)rep,,'P eK, (1) F. (1).*332-241.)D Prop *332-243. F: KE FMconx. f! P.P CIrS'U1"K. i.ep,,'P =I s"K [*332-24. *330-43] *332-244. F:. c FM conx -1I.)D Dem. F. *331-26. *.330-43.:) F:. Hp. D: 8K, + 1 Al" [*332-13] D repk"P I Ir SWI"'K.. ft! P (1) FI- *332-11.:)F:. Hp.:):repc'P=IrS6(166K.).PC'Irs'P'6K (2) F.(I). (2). *332-243.:)F. Prop R. &W. III. 24

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370 QUANTITY [PART VI *332-25. F:KceFMconx.~f!P.ft!rep/,,Q.PC-Q.).

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repK,'P=repK'"Q Dem. F. *332-11.) F: Hp..P Crep/,Q(1 F.*332-22. DF:Hp. D. rep,,'Q E K, (2) F. (1).(2). *33 224.) F. Prop *332-26. F:KeFMconx.ft!PtAQ.ftrep/, P.ftjrepK6Q.:). rep/PT = rep/,Q = repK1'(P A Q) [*332-25] *332-27. F:1KeFMconx.ft!P.ft!rep6Q. -f! Q A repK/P.) rep/,P= repK/Q Dem. F. *332K11)F: Hp..Q Crep/Q. [lip]).! rep/,P A repK'Q (1) F. *3.32-22.)F H lp.).rep/lP, repK"Q E K, (2) F. (1). (2). *331-42.)F. Prop *332-31. F: KeFMCOflx.i, ME K,.). rep/,(LIM) EKI, [*330'611. *331-4712. *332-23] *332-32. F: Kc e FMcoinx. L, ME K1.:). repK/(L I M) = repK (M I L) [*330'611. *331P471.2. *332-24] *332-33. F: K c FM conx. repK/P, rep/,Q C /4. ft! P IQ.). repK/(P IQ) = repK'{(repK"P) I(repK'Q)} = rep/{l(repK/P) IQ} = rep/PK ~ repK'QJ Dem. F. *330'6 *3:31-12.:) F: Hp. ft! repK/ P.-! rep/Q. [*332-11]).P C rep/'P. Q C rep/,Q. (1) [Hp])t! P IrepK"Q(2 F. *330-6. *332-31. (1).)D F: Hp. D. P I repK/Q C rep/lP Irep/,Q. ft! rep/{"repK/P Irep/,QI. [(2).*332-25]:). rep/,(P Irep/,Q) = repK'IrepK6P I repK/Q}. ft! rep/"(P IrepK'Q) (3) Si milarly F: Hp. D. rep,, { (repK P) IQ} repK'{(rep/,lP) I(repK'Q)} (4) [Hp.(3). *332-25]).repK/(P I Q) = rep/,(P IrepK6Q) (5) F.(3). (4). (5). DF.Prop *332-34. F: Hp *332-33.:). repK'(PI Q) e K, [*332-3133] *332-35. F:KcFMCOnx.L,MN, NK,.:). rep,, '(L IMI N) = rep. ItL IrepK/(MI N)} = repK'[{rep/,(L I M)} NV] [*330,613. *332-31-33] *332-36. F:Hp *332-35.).repK/(L IMJI N) e K, [*332'35-31]

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SECTION B] ON THE REPRESENTATIVE OF A RELATION IN A FAMILY 371 *332-37. F-: KEFM conx.repP, repk,'Qrepk'"R E1,.f! PJQ IR.) rep,,'(P IQ 1?R) = repK'6trepk'lP I rep/,Q Irep,,'R} = rep/tllrep,,'P IrepkR IrepkQ} Dem. ~~~~~= repic'frep,,'Q Irep,,'R Irep/,PI I-.*332-33.) I- Hp. D. rep,,'(P IQ IR) = rep/t"rep,,'P IrepK,'(Q IR)} [*332-33] = rep/icrepK'P Irepclrepc'Q Irep/"R)} (1) [*332-35] =repc IlrepK'P I repc'Q Irepc'R} (2) F- (1). *332-32. D F- Hp. D. repc'(P IQ IR) =rep,,'{repc'lP Irep,,'(repK'R Irepc'Q)} [*3,32-35] = repc'{repK' ec Rre~Q 3 F-. (1).*332-33-32. D FI: Hp. D. repc'(P IQ IR) = repc6trepc'(repclQ Irepc'R)} Irep~'cP] [*332-35] = repK'{repK'Q IrepRI? repc'P} (4) F-. (2). (3). (4).)IF. Prop *332-41. I-:.KEFMconx.L,MlNeK,.): rep,C(L IM) = rep,'(L IN). ~.M =N Dem. F -. *34-34.) F:Hp. repc'(L M) =repc'(L IN).)D. L Irep, '(L M) = L Irep/ l(L IN). [*332-35] D). repK/(L IL M) = repK/(L IL IN). [*330-31])D. rep/M = repK'N. [*332-241] D).M=N:) F-.Prop *332-411. F-:. KeEMconx.L,MNEK,.:):repK'(MIL)=repK'(N IL)..M N [*332-32-41] *332-42. F-: K eFMconx.LMEK1.). Cuiv'rep~'(L IM) =repl'(L IM) [*332-32-15] *332-43. F-:.KcFMcotix.L,MNlEK,.D: N=repK'(L M).. L eK(L = repK =rp'M N).. M= rep.c'(N L). z. M =repK'(L N) Dem. F-. *332-35. *330-41.)D F-: Hp. N= repK/(L Al).:). repKI'(L IMl M) = repKf(N' M). [*330-31] D). repc'L =repK (N Al). [*332-241]).L = repK'(N IM).(1 [*332-32.*330-41] D. L = repK'(M IN) (2) F-. (1). *330-41. DF-: Hp. L =rep/"(N iM).)D6N= repK'(L JM) (3) F-. (1).(2). (3).)4-Prop 24-2

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372 QUANTITY [PART VI *332'44. F:.K6FjIconx.L,MNEK,.:):repK'"(LIM)=N.=-. LIME-N [*330,6. *332-24-31] *332-45. I:-:Hp *332-44.)repK'(L IM) = N. =. repK'(L 3I M N) =Irs'CU'6K Dem. I-. *332-35.:) F:. Hp.:): rep,,'(L IM) =N.:). repK'(L IM IN) = rep,,'(IVN N) [*332:24.*330-31] =I rs'P' (1) F-. *332-35.) I-:. Hp.): repK'6(L IAI N) =1 r' s'(1""K.. repK'[{rep/I(L IM)} N] =Ir A 'O"K. [*332-31-43]).repK'(LIM) =repK'N [*332-241] I-.(I). (2).) F -.Prop -N (2) *332-46. Dem,. I-:. K E FMconx.L, Mec,.):LIM CI. =. L = M F-. *33043-611. *.332-243.) 1- Hp. L IM C I.). rep,,,'(L IM) = I r s'(I"',c. [*332-43.*3:30-43] D. L = rep,,'M [*332'241. *330-41] =Mil F. (1).(2). DF.Prop (1) (2) *332-51. F:KeFMconx.P,Qcx.:). rep,,'(PIQ)=Q P Dem. F. *3.31 24. *332-32.) F: Hp. D. rep,,'(P IQ) = repKI'(Q 11P) [*332-241] = Q IP: D F. Prop *332-52. F:/ceFMconx.P,Q,R,ScC.:). repK,'(PIQIRIS)=QIS PIR Dem. F.*330-613.*331V12 124.) F:Hp.). t! (P IQ) I (RfIS). [*332-33-51]:). rep,,'(P Q R S) =rep,,'(Q P 8 R) (1) F.*33O561P61.)F: Hp.:).QIPISiRC-QIP~.!~II 2 F. *331P52.)F:Hp.:). QjSI PI RE, (3) F.(1).(2). (3). *332-24.) F.Prop

SECTION 13] ON THE REPRESENTATIVE OF A RELATION IN A FAMILY 373 *332-53. I-:,ceFMconx.P,Qe/K.peNCinduct.).rep '"(PIQ)-P=PIQP Dem. F. *330-624.:)F: Hp.:). fj!(IflQ) (1) F.*330-73.:)F: Hp.:).(PtQ)PC-PPIQP (2) F.*331-53. DF:Hp.:). PPIQP6K, (3) F. (1). (2). (3). *332-24.) F. Prop *332-61. F Kc e FM conx. L e KL. rep,,"Potid'L C Kc, Dem. F. *332 243. *330-43.) F: Hp.). rep,,'(I r CIL) e Kc, (1) F. *332-31.. F: Hp. Me6 Pot'L. repk'"M e K,:). rep,,'tL Irep"'M} e K, (2) F.*330-624.) F: Hp.-Me Pot'L. D. ftL IM (3) F. (1).(4).JInd tit.)F.Prop *332-62. F: KeFMconx.A'%jePot'P.ft!rep,,'P.:). repK'"Pot'P C rep/'IPot'rep,,'P Dem. F. *332-242 F): Hp. rep,,'P = repW'"repK'"P (1) F.*332-22.:)F:Hp.:).rep,,'Pe,6 (2) F. (2). *332-61.) F: Hp. Q ePot'P. rep,,'Q erep/'11Pot'Irep/,P.).rep,,'QeCK, (3) F.*91-36.)F:Hp.QePot'P.).ft!PjQ (4) F.(2). (3). (4). *332-33.) F: Hp (3)).repk'"(P IQ) = repK'{jre p/P IrepK'QI. [Hp.*91P36]:). rep/,(P Q) erep/,"'Pot'rep/,P (5) F. (1). (5). Induct.:) F. Prop *332-63. F: Hp *332-62.:). rep,/'Pot'P C Kc, Dem. F. *332-22.) F:Hp.D. rep/,P eK, (1) F. (1). *332-62-61.:) F. Prop *332-64. F:K e FMconx. rep,"'Pot'P C K,.). rep/'1Pot'P CrepK'6Pot'rep/,P Dem. F.*331P26.*332-13.)F:Hp. K'E.I.).Ar —IEPot'P (1) F. *330-6. *331-12.:)F:Hp.). A erep/',6Pot'P (2) F. (1).(2). *332-62.)F:Hp. K,-,E1.). repK/'Pot'P Crep/'6Pot'rep/,P (3) F.*330-43. *331-22.-:). F:Hp. KeI.) IC, = t'(I rS'P'IK) =K (4)

374 -.QUANTITY [PART VI [*332-243-13.(4)]).rep,/P = I r s'tI '%'C (6) [(6).*332-241]).rep,,"Pot'P = t'Irep/,Irep/,IP (7) F.3) ().).Prop *332-65. F:A E 6Pot'P.!r rep/'P. Pot'P C s'R~IPot'rep/,P Demn. I-. *332-11.)!-F: Hp.)D. P C rep/'P (1) F.(1).) F:Hp.Q~ePot'P.RePot'Irep,,'P.QC-R.).QIPC-RIrepK/P (2) F.(1).(I2).JInduct.)F.Prop *332-66. F:ft!repK,'P.REPot'repK,'P.:).(HQ).QEPot'P.QC-R [Proof as in *3.32-65] *332-67. F:/ceFMconix.A~ePot'P.ft!repK'"P.:). repK"Pot'rep,,P = rep/'"Pot'P Dem..

F. *332'242.)F:Hp.).rep/, rep,,/P = rep/ P (1) F.-*332-66.)F:. Hp.: Re Pot'rep/, P.).! R IP (2) F. *332-22.)F:Hp.).rep/'PE6, (3) F.(3). *332-61.)DF:.Hp.)D:R ePot'rep/,P.).rep/,R E K, (4) F..(2). (3). (4).*332-33.) F:. Hp.:): R e Pot'rep/,P.:). rep/,(rep/,RI? rep/,P) = rep/,(I?1 repK,'P) (5) F. *332'33. F: Hp. R? e Pot'rep/,P. Q E Pot'P. rep/I? = rep/,Q.) repK"(Q IP) = rep/,(rep/,RI? repK/P) P(5) = repK/,I(R Irep,C'P) (6) F. (6). D F: Hp. R e Pot'rep/,P. rep/I? e rep/,'Pot'P.) rep/,(R Irep/,P) e rep/'"Pot'P (7) F. (1). (7). Induct.:) F: lip.:). rep/'Pot'rep/"P C rep/'"Pot'P (8) F. (8). *332-62.)DF. Prop *332-71. F: Ce FM corix. LME K,. rep/'"Pot'(L IM) = rep/"'Pot'rep/"(L M) Dem. F.*332-31.*330-6. F:Hp.). f IrepK/(LjIM) (2) F. (1). (2). *332-67) F. Prop

SECTION B] ON THE REPRESENTATIVE OF A RELATION IN A FAMILY.37 5 *332-72. F:Hp *,332-71.).repR,"Pot/(LIM) C/K, [*332-31-6171] *332-73. F K E FlconX. L, ME /,.). Pot'(L IM) Cs'R1"'Pot'rep,,'(L M) [*332-65 31. *330-626] *332-74. F: KeFMCOnx.LMElK,.PIEP~t'M`.). Dem. ~~~~~repK,(L I P) = repK,'(P IL) = rep,,'(L IrepK"P) F.*330-627. *332-6133.) F Hp.:). repK/(L IP) =repK'fL I rep/ JfPJ (1) [*332-61-32] = repK/{rep./P ILI [*330-627.*.332-6133] = rep,,'(P IL) (2) F. (1). (2).:) F. Prop *332-75. F Hp *332-74. D. A! rep,,'(LI P) [*332-74'61-31 *330-6] *332-8. F:K eFM conx.LIM6 K,..4-NC ind. D. Dem. F. *332-243.) F. *301P21. *332-33. *330-626.)D F:Hp. rep,, (L IM)t = rep,,'(L I Mt).) repK(LM)t +~I=rep,,,LJ IMt L IMI [*332-37] = repK{L ri(~)M [*332-32'33] rep/f[I rep/,(L I LIP) I MI [*332-37] =rep,,c~ II M~+cl} (2) F. (1). (2).JInduct.) 3F. Prop *332-81. F:Ke FM conx. V,0-eISNC ind -t'0.LIE K,.D. rep, iLvxC0 = rep#/I(repK'LI')lI Dem. F.*301V23.:) F Hp. rep/ILvxc0 = rep/"(rep,/Lv)ao.) repK cLvxc(o'+cl) =rep~c'(LvXca I LI) [*332-33] = rep,,l't(rep/lLlv)0' repK'Lv}I [*301 23] - repK (repK/Lv)a+cl(1 F. (1).Induct. F. Prop *332-82. F: K eFMconx. v eNC ind -t'0. L, ME6K,. repK/(L IM)v =rep/{Irepc/(L IMI. Dem. F. *.332-33.D F: Hp. repK/(L IMi)v = {repK/(L IM)1v.) rep,,'(L IVM)v+cl - repc'[{rep/I(L I M`)}v' repK'(L IM`)] [*301P23] = repK/{rep/"(L IM)}v,+ol(1 F. (1). *1 13-621. *301-2. Induet. D F. Prop

*333. OPEN FAMILIES. Summary of *333. An "open" family is defined as one such that, if L is any member of Kc, which is not contained in identity, then every power of L is contained in diversity, i.e. Lpo C J. We shall often have occasion, both in this number and later, to consider the class K, - Rl'I, and in later numbers we shall often have occasion to consider the class - Rl'I. We therefore put *333'01. e a= X-Rl'I Df *333-011. ca = (1,)a Df Thus ca consists of all members of K, which are not contained in identity, i.e. (if Ic is a connected family) all members of c, except I s' I"Kc. The definition of an "open" family is *333 02. FM ap = FM n 2 {s'Pot"c,a C RI'J} Df From the point of view of the application of ratio, the hypothesis that a family is open is very important. To begin with, it insures (*333'18) that K,a consists of "numerical" relations (cf. *300), so that if L e K,a, we have Pot'L = fin'L (*333'15), and in virtue of *300-491, the existence of open families implies the axiom of infinity (*333'19). Again, in an open connected family, if L, M are two different members of K,, all the powers of L M are contained in diversity, and therefore the representatives of these powers are members of,ca; that is, we have *333-22. F: Ke FM ap conx.L, M e,. L M. D. rep,"Pot'(L M) C fa It follows from this proposition that, with the above hypothesis, if a is any inductive cardinal other than 0, La MC is not contained in identity, and therefore L' + M' and rep.,'La + rep'Mc. Hence by transposition we obtain the two propositions: *333'41. F:. K eFMap conx. L, M e c,. o e NC ind - '0.: rep/'LO = rep,,M. -. L = M

SECTION B] OPEN FAMILIES 377 *333 42. F:. Hp *333*41.: L = M7.-. L = M Hence we obtain *333 43. F:. Hp *333341. D: [! La Ma. L-M This proposition shows that in an open connected family, no two members of c, have the ratio 1/1 unless they are identical. Again it follows from *333'41 that if LPXCT and MaXCT have the same representative, then LP and MT have the same representative, and vice versa, i.e. *333'44. F:. e FM ap conx. L, Me K,. p, a, T e NC ind - t'0.: rep,'CLPxT = rep,'MoXC. -. rep/'LP = rep,'MA Hence we obtain two propositions which are vital for the application of ratio, namely: *e333'47. F:. K e FM ap conx. L, M e,. p, e NC ind - tO.: rep'LP = rep,'AM. -. [! LP A M' *333'48. F:. e FMapconx. L, M ec,. p, a, eNC ind - 'O.: X! LP A M. -. a! LpxT A MOXcT On comparing this last proposition with the definition of ratio (*303'01), it will be seen that, whether p is prime to ao or not, L has to M the ratio a/p when, and only when,! LP A MA, i. e. (by *333'47) when, and only when, rep'LP = rep,'Ma. From *333'47 it follows also that, if M e /,a, MP and MO will not have the same representative unless p = a (*333'51), i.e. *333*51. F:. e FM ap conx. Me,a. p,o e NCind.: rep,'MP = rep,'M.. p = - From this it follows that no member of Ka has any other ratio to itself than 1/1. Again, by *333'47'48'51, we have *333353.: K e FM apconx., M e, a.! LO MP.! L MI.). /L Xooa = Xop Hence if L and M have the two ratios p/a, pi/v, we have p/a- = p/v; that is, no two members of Ga have more than one ratio. The applications of ratio indicated in this summary will not be made till the following Section; they are here mentioned in order to show the utility of the propositions of the present number. *333'01. K K = - R'I Df *333'011. Ka = (K,)a Df *333 02. FM ap = FM n {s'Pot"a C Rl'J} Df *333'03. FM ap conx = FM ap n FM conx Df

378 QUANTITY [PART VI *333-1. F:MEKCa.=-.(3[P,Q).P,QEtc.M=PIQ.ft!MAtJ.. MeI. a!M J[(*333-01 01 1)] *333-101. F:.KcEFMap.=:KcEFM:MEK~a.PEPOt'M.)MP. PCJ::KeFM:Me 1ca.)DM. Mpo Cf [(*333-02)] *333-11. F:,ceFMap.LeK~a.).LC-J. L2C.J.LA~L=A.L+L.fj!L [*333-1101] *333-12. F:KeFMapconx.ftIrepc'P.t!Pi-J.:).

rep/,P E Ic~a. (rep/,P)po C J Dem. F. *332-11.:)F:Hp.:).! rep/,P AJ. [*332-22. *333-1] D. repP eT K~a(1 F. (1). *333-101..)F. Prop *333-13. F:KeFMapconx. ftjrep/"P.ft!PtJ.).P,0 C-J Dem. F. *332-11.)I-: Hp.D. P C iep/,P(1 F. (1). *332-22.) D F: Hp. D. ft! (rep/,P) A J. PPO C (rep/,P)p0. r~ep/,Pe E14 [*333-1]). PP0 C (rep/"P)p0. rep/,P e K~a. [*333-101] D. P0 C J:)DF. Prop *333-14. F: KeFMapconx.L,MEK,.L+M.:).(LIM)poC-J Dem. F.-*330-626. D F: Hp. D.A e Pot'(L IM) (1) F.*332-31. *330-6.)F:Hp. D. f rep/'(L IM) (2) F.*332A46.TranspD F: Hp. D. f! (L IM) AJ (3) F. (1). (2). (3). *333-13. D F. Prop *333-15. F: e FM ap. L /c ~). Pot'L =fin'L =finid'L -t [*121P501. *333-11-101] *333-16. F:KleFMap conx. L, MexIc. L +M.) Pot'(L I M) = fiti'(L IM) = finid'(L IM) - tf(L I.M, [*121-501. *333-14] *333-17. F: i cFMap conx. t! rep/,P.!P AJ.)D. Pot'lP=fin'P=finid'lP-t'lPo [*121P501.*333-13] *333-18. F: Kc E FM ap.). tc~ C Rel numn [*333-101. *300-3] *333-19. F:iKeFMap -tiftA.D). Infin ax [*333-18. *330-624. *300-491] *333-2. F: a! FMap conx. D. Infin ax [*3.33-19. *331-12]

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SECTION B] OPEN FAMILIES 379 *333-21. 1- E FMap conx.L6 EK~a. rep,,"P ot,'L C Kja Dem. F. *332-61.) Hp..rep,,""Pot'IL CK, (1) F.- *333-101. *330-624.) I-:. Hp.) A Pot'L. Pot'L C RII'J: [*332-11.(l)] D:Merep,,"1Pot'1L.:).! MA J (2) F.(1).(2). *333L.)I- DF. Prop *333-22. I-: FEFMapconx.L,Me6K,.L[M.). repK"6Pot'(LIM) CK~a Dem. F. *332-71.)FHp.). repK"Pot'(L IM) = rep,,"1Pot'repK/(L IM) (1) F. *332-46-11232-31. D F Hp. D. rep/,(L IM) e Kle' (2) F.(1).(2). *333 -21.)F.Prop *333-23. F:KEFMapconx.A~EPot'P.ft!rep,,'P.ft!PAJ.:). repk"Pot P C IKla Dem. F. *332-62. F F Hp.). rep/',Pot'P C rep,,"Pot'1repK/P (1) F. *332-1122. *333-1. F F: Hp.. repK'"P e 1c (2) F. (1). (2). *333-21. DF. Prop *333-24. F:KeFM conx Are Pot'P. A!rep'PK6-Ve NC ind a! v A t2',R. Dem. rep,, P = reP,,(rep,,"P)v' F.*301-2. *332-243.:) F: Hp.:). rep,,'PO =1I r S'P1'K = rep/,1(repK,1P)o (1) F.*332-63. *330-6. *301-16-22.)D F: Hp.)D. rep., Pr, repc'P E tc. - f! P +1*(2) [*301-21. *3M2'33]:) rep,,6Pv+c1 = rep,,61(rep,,'P") Irep,,'P} (13) F.(2). (3).:) F: Hp. rep,, P v = rep,j'(repc LP)v.). rep, 'Plv+c1 = repK,1trep,CL1(rep,,6P)v' rep,,'P} rep,,'(rep/,P)v, rep,,'P e K, (4) F.(2). *330'624. *301 21. D F: Hp.). ft! (repK'P)" repk'P (5) F. (4). (5). *332-33.-) F: Hp (4).).rep,,'Pv+cl = repK'{(rep/'P)v I repK'PI [*301 '21] = repC1L(reP,,1P)v+cl (6) F. (I). (6). Induct. D F. Prop A hypothesis equivalent to ve NC ind. a! V A t2,'1? is v e C' U ~ t3'R. It is sometimes convenient to substitute thi's for the other. *333-25. F:xeFM conx. L, M e,.v eNCind.a2! VA t2LL.) Dem. ~~~~~~~~~repk '(L IM)v = repK'{repK'(L IM) J,, F.-*330-626. *331-12.:) F:Hp.)ve Pot'(L IM) (1) F.*332-31.*330-6.:)F:Hp.:). ftrep./(LjM) (2) F.(l).(2).*333-24.:)F.Prop

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380 QUANTITY [PART VI *333-32. ~:EM~XLMK,T I1(U~tcL L c Dem. F -.*330-61. *301-2. DF:Hp.). D!LOjMO (1) F -.*330-623. F-:. Hp.D: SECK. Ds. SILP IMaC. LP IMr IS: (2) I- (1).- (3). (4). Induct.) D -. Prop *333-33. F-:KcEF,4conx.L,,MElc,.

a-e[I'(U~t3'L).:). Dem. ~~~~~~~~~~repc(L,7 M`7) -rep.'(L M, I -.*333-32. *332-243.)D F- Hp.). re PK"(LO MO) = I r s'Oi"xc = repKI(L IM)o(1 I -. *332-37. *301-21.)D F-: Hp.). rep,,'(Llr+cl1 Mdr+cl) - rep '[rep,,'(Lr IfMO) IrepKIL Irep~'M} (2) F-. (2). D)F: Hp.re p,'(Lr Alm) =repK'I(L IM)O. D. repK/(L0'+cl IM'+,,') = rep,,'{repK'(L IM)IT repK'lL Irep/lM~J (3) I - (3). *333-32. *332-37.)D I-: Hp (3).) D. rep~ll(La7+cI Ma'+c) = rep/{I(L IM)O'l L IMI [*301-21] = rep/( )Kc (4) F-. (1). (4). Induci t.:) t~.Prop *333-34. F-: Hp *33333.). rep,,'(La, Mr)= repK'trepK'l (L1M`)}lrrepK'(LjIM)lr Dem. F -.*330-626-6. *332-31.)D F-. (1). *333-24. D F-: Hp. D). repK/{rep/I(L M)J0 = repcl'(L IM),' (2) F-.(2).-*333-33.)-. DProp *333-41. F:-:.ceFMapconx.L,MEKc1. a-CNC ind -— tO.)D rep~c'Lo = repK'MM.. L =M Dern. F-. *333-34-22-2.)D F-: Hp. L =[M.). rep/l(LU A-7f) EKiy. [*333-21-32.*332-33] D. rep/{frepK/Llr repKM'Mr e KLa. [*332-44.Transp])., t-~rep,,'Lo' repc'aP C. I r s'U(i4.,C [*332-15-46.Transp] D. repc'1 4Lr rep 'MO'(1 F-. (1) - TraDsp.) F-. Prop *333-42. F-:.Hp*333.41.:):Lo7=MY.=-.L=M [*333-41]

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 333-43. F:.Hp.333-41.:):fj!L~tAMT,.=.L=M Dem. F.*33.321.*332-26.)F:Hp.ft!J~AM'.:).rep,,LTw~rep,,,APo. [*333-41].L= M (1 F.(1). *330-624.DIF. Prop *333-44. F:.KceFMapeonx.L,Mc6K,.p,o-rieNCind-LfO.:): Demn. repK'LPXCT' = rep,,~XT repK'2 eK F.*301 5.*333-24.) F:. Hp.:): repK/LpxcT = repK'fMrxc.repK'(repK~P) repKI(rep,,M a' [r*333-41-21] repK'LiP = rep, 11V'M.) F. Prop *333-45. F: Hp *333-44.:):LPXcTr-McTXC). rep/, Lp=repc'M,7 [*333-44] *333-46. F:.Hp *333-44.:):f! Lp~xC? A JrXCr.).repK'LP = rep 'Mu' Dem. I-. *332-26. *333-21. F: Hp. A! LP XCT A Mo'xCTr). repK(LPx c -= rep,'M~xcTr (1) F.(1).*33344. DF.Prop *333-47. F:.KeFMapconx.IL,MEKc1.p,o-, eNCind-t'O.): Dem. - ~~~~~~~rep,,'Th = repc'Mcr t LiP A Mc F.*333-46.:)F:Hp.t! LPAMcT.) rep~cLP~repgc"Mr (1) F.*332-53. *72-92.) F: Hp.P, Q, R, Se ic. L = P IQ. M = RB S.:). LiP = (PP IQP) r (1'LP. M(=(R IS~r) r M(" repK'LP=PPjQP.rep "Mo'=Ro'ISo (2) F. 2) 3514.) F: Hp (2). repK,'LP - rep '1JP.. LiP A Mo, = (PP IQP) r' (PILP v, OilPMc). [*330-72] D. t! P A MO (3) F.(I). (3). D)F. Prop *333-48. F:. iccFMapconx.L,Mcic1.p,o',rcNCind-t'O.):!!LP A MT.~ ft! LPxcTrAATXC7 Dem. F.-333-46. DF:Hp. f! LP A r.) rep' ILP =repK'Ma' (1) F. *330-624. *332-61.)F: Hp.). A, PotLP. f! rep,,c'P. [*333-24]:). repKILP~c -,r repKII(rep,,"P)Tr (2) Similarly F: Hp.).rep, IMO'xc -= repKI(repIMr)r (3) F -(1) (2) (3)) F: Hp. ft L A Mif.. repK,, LpXcr =repc CMO'Xc7 [*333-47] ft! LPxCrA MorXC (4) F*333-4647. DF:H. AL~rAM7c.:.f LP (5)O

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 *333-49. I-: /eFMapeonx.L,MeK,.p,o-eNCind -L'O. repK'PI.=repK"'AP. Dent. F. *333-21. *330-6.) F Hp..!rep,'LP. [*332-11]:).P C rep,,'LP. [*72-92] D. LP = (rep,,'IP) r 'LPL (1) Similarly F Hp. D. J10= (rep,,'Mo) r GPMO. [lip] Mc' = (rep/,LP) r U'MMQ (2) F. (1). (2).:) F: Hp). LP r GPM'= (rep,,'Lp) r (U'P n (I'Ma')=Mr (fU'P (3) Similarly F: Hp. D. (D'"M~) 1 LP = (D'LP) 1

Mr (4) F. (3). (4).)D F. Prop *333-5. F:.tce FM apconx.P, QE/Ic.oa NC ind- t'0. Ta= Q,7.=t ~[P AQTEP = Q [*333-42-43. *331-24] *333-51. F:.KEFMapCODX. MeK~a.p,oaENCind.:): repKM =MerepK'Mr - p0 Demn. F. *333-47.)D F:. Hp.- rep/ AI' - repK 'AM. D MP A MO': [*301P23.*120-412-416]:): p ~a.). f! MP-c17A f. [*333-101])p = O (1) Similarly F:. Hp (I).):a-, p. D. p a (2) F. (1). (2).)F. Prop *333-52. F:Hp *333-51.)D:MP = Mr..p=a [*333-51] *333-53. F:E~poxLM KL t ~ P t vAM,.D AP Xe0 a= X0 p Dem. F. *333-48. *301-16.) F: Hp.) t~! Lgxc~r A MIuxcP. ft! LvxcP A Ml-~xcP. [*333-47]).repK9 LIxc- rep, IM/- xc -~P repKcLvxcp. [*333..51] x.,cxa = vx,,p:)DF.Prop

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*334. SERIAL FAMILIES. Summary of *334. The purpose of the present number is to consider what properties of a family K will insure that 'Kca is serial, or has one or more of the properties characteristic of serial relations. Suppose, for example, that K consists of distances on a line. Then Ka consists of those distances which are members of K and are not zero. Any selection of distances on the line may constitute AC; thus e.g. K may consist of all distances which are integral multiples of a given distance, or of all which are rational multiples of a given distance, or of all distances from left to right, or of all distances on the line in either direction. It is plain to begin with that if s'Ka is to be serial, K must not contain equal distances in opposite directions, since if it does, (h'Ka)2 will not be contained in diversity, i.e. s'Ka will not be asymmetrical. We call a family K asymmetrical when no member of Ka has a converse which is also a member of Ka. The definition is *334-05. FM asym = FM v^ n (K n Cnv"c C Rl'I) Df It will be observed that s'Ka C J in any connected family, by *331'23. If Ec e FM asym, we have also ('Kca)2 C J. In order to secure that e'Ka shall be transitive, we require that the field of K should contain at least one "transitive point," where a " transitive point" means a point a such that any point which can be reached from a by two successive non-zero steps can also be reached by one non-zero step, i. e. such that (si'a)"s'Kta'a C s'Ka'a. The definition of transitive points is *334 01. trs'K = sI"K n {(S'Ka)",'Kaa C S'Ka'a} Df Thus if a is a transitive point, and R, Se ca, there is always a member of Ka, say T, such that R'S'a = TPa. It will be seen that if K is a connected family, the existence of a transitive point implies that the family is asymmetrical. Again, if there is a transitive point in a connected family, then R, Se Ka.. R S e ca, by *331'32; hence Ka is a group. The converse also

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384 QUANTITY [PART VI holds, i.e. if Ka is a group, any member s'a([cN is a transitive point (*334'11). Hence if there is any transitive point, every point of S'(I"Kc is a transitive point. The definition of a transitive family is *334-02. FMtrs = FM n K (2E! trs'K) Df By what has just been said, a connected transitive family is one in which Ka is a group, i.e. *334'13. F:. e FM conx. D: e FMtrs. S. s'cA "ca C ca A connected family is transitive when, and only when, S'/c is a transitive

relation, i.e. *334'14. F:. K e FM conx. D: K e FM trs. -. sKs e trans In order to secure that 'Kca shall be a connected relation, it is not enough that K should be an FMconx, i.e. that s'I'CKc should have at least one connected point. We require that every point of S'(e"K should be a connected point. This will be secured if there is a connected point which belongs to the field of every member of K,, i.e. if a! conx'cK np'C"K,. For suppose a e conx'"K A p'C"c,. Then if L e K,, either L'a or L'a exists, and is of the form R'a or R'a, where R eK. Hence, by *331'32, L is identical with R or with R; hence K, = K v Cnv"K. Hence by *331*4, h'Ka e connex. Conversely, if eEFMconx and s'cae connex, it follows from *331'32 that K, = KV Cnv"K; hence p'C", = s'("K, and therefore we have! conx'Krnp'C"Lc,. Hence putting *334-03. FMconnex = FM ^ (! conx'K np'C"i/,) Df where " FMconnex" means "families having connexity," we have *334'26. F:. c e FM conx. K: c e FM connex. -. 'ca e connex.. K, = K v Cnv"K. -. C6"K, = (aI" and *334'27.. FM connex = FM n K (s'I"K = conx'K. E 4 t'A) I.e. a family having connexity is one whose field'consists wholly of connected points and is not null. We thus secure (1) s'ca C J by the hypothesis K e FM conx, (2) s'ca e trans by the hypothesis K e FMconx n FM trs, (3) sca e connex by the hypothesis ce FM connex (which implies K e FM conx). Hence we secure 'Kae Ser by the hypothesis ceFMftrs n FMconnex. Whten this hypothesis is fulfilled, we call Kc a "serial" family; thus we put

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SECTION B] SERIAL FAMILIES 3 8 -5 *334-04. FM1 sr = FM trs A FM connex Df and we have *334-3. F:Ke FM sr.)ai'ce S er *334-31. F:.KE6FM.Irrs'GI"K1ieK.):K FM sr..,'KaE Ser - 'A An important special case, which is briefly considered in this number, iis the case when the domains of members of K are the same as their converse domains, i.e. when DC = (I cc. This case is illustrated, e.g. by the family whose members are all relations of the form (+g X) ~ C'H9, where X e G'H'. It is also illustrated by cyclic families, which are considered in the next Section but one. When D",K(I" K) if K is a family, SO is K u Cnv,"K (*334-4), and if K is a connected family, so is K v Cnv"K (*334-41). In the case of the above family, whose members are (+g X) ~ C'1Hg where X E ORH, K v Cnv"K will consist of all relations (+g X) ~ CG'Hg where X E C'1Hg, i.e. it will consist of all additions of positive or negative ratios to positive or negative ratios. A connected family in which D "K = UI"K is a family having connexity, i.e. *334-42. F K E FM conx. D"K = UI"K. K FM connex The definitions and propositions of this number are much used throughout the remainder of Part VI. *334-01. trs'K =S'tI"CK A a^ {(h'Ka)"8h'Ka'a C h'K'a}a Df *334-02. FMtrs = FM nk (2[! trs'K) Df *334-03. FM conn ex = FM A k (a! conx'K A p'CG"K,) Df *334-04. FM sr = FM trs n FM connex Df *334-05. FM asym. = FM A 'K (K A Cnv"K C Rl') Df *334-09. F K C FM conx.). h'a J [*331P23] *334-1. F:: KeFM.>:. a etrs'K.= a e s'UI"K R, SE Ka. - R, S (jT). TOE Ka. R'&~a =T'a [(*334-01)] *334-11. F:. K e FM conx.)a e trs'K. aC ES~O'K. S"KaI "6Ka C Ka Dem. F.*331-33-24.:)F:Hp..R,S6Ka.:).Rj8E K, (1) F.(1).*331'32.:) F:Hp.TEKa.-R'S'a=T'la.).RjS=T (2) F.(2).*334-1.)F::Hp.:):. ae trs'K. ~E S'Ul"K: R, SE16Ka R,. (gT). TE16Ka.1? S = T/U [*13-195]:aE6S'O"K:R, SEKa.)RS. R I S EKa:)F Prop R. & W. III. 25

University of Michigan Historical Math Collection: Principia mathematica, by Alfred North Whitehead ... and Bertrand Russell.

386 386 ~~~~~QUANTITY [ATV [PART VI *334-12. F -K EFMconx.a, x ES'I"6K.) D: (t E trs'K. E. x e tI s'K..- 8 'Ka3 Ii C Ka [*334-11] *334-13. F:.KFEFMCOiix.:) KEFMtrs.= E.S'rK"~Ka C Ka [*334-12. *331-12. (*334-02)] *334-131. F:K E FM conx n FMtrs. 1E~Ka.).Pot']?C Ka [*334-13. Induct] *334-132. F K EFM conx r^FM trs.). sPOt""KC K [*.334-131] *334-14. F. Ke FMC~l cn KIEFM trS = 'Kae traDS Dem. F. *4151. *334-13.:)F:.llp.)Ke FM trs. D.('a2.~ (1)?, ScKa.x es'"Wxm.:)R,s,x. (gT). Te6Ka. R'S'x= T'x. [*331-31'33-24]:)R, S, x. (a[T). TE6Ka. R IS= T. [*13-19.5]:)R,S,x.]?I SC Ia (2) [*334-13])Ke FM trs (3) F. (1).(3).) F. Prop *334-15. F:KeFMconx^Fl~trs.).S'K"ICK= K Dem. [*50-62-63] D):SE6K.)R IS, SI1RE K (1) F. (1).*334-13.)F: Hp.)D-S'K "'KC K (2) F. *331P22 *50-62-63. F F: Hp.)D. K C S'K "' (3) F. (2). (3). F. Prop *334-16. F:K eFM conx FMtrs.1?E Ka.). JpC-J [*334-131-09] *334-161. F: KeFM conx niFM trs.Re6Ka.aC S'f"'K)R*lae No [*334-16. *123-191] *334-162. F: a! FMconx nFMtrs-1l.).Infin ax [*334-161] *334-17. F: KeFMconx ni..). Ka= A [*331V22] *334-18. F: K FM conx -1 CY~a = 8'P'K = CC8'UKa. [h'K& Ic!1a Dem. F. *331P22-321.)F:. Hp) -: a! Ka: [*330-52] D: a ES'(Ic"K. (a11).]? C Ka. ae EU'] [*40A4] D. a C S'OI","K(1 [*41A45]:). aEG"K~, (2) F.(1).(2). *331-12.) F.Prop

SECTION B] SERIAL FAMILIES 387 *334-19. F KEFM.). C'K'aC8'G '1K [*41-4. *330-52] *334-2. F::. KeFM.):aep'C"KL. =:. LEK,.)L:E! L'a. v. E!L'a [*330-52] *334-21. F K EFMconnex.).,= K VCnV"K Dem. F.*334-2. *331 11.) F:.Hp. a6CoDX'KrP'C"K,.LfK,.:): (M[R):1 E K v CIV"'K: L'a = R'a. v. L'a = Ra [*331-42-24])(aR): RE6Kv Cnv"'K:L=I?.v.L~=R (1) F. (1). *331 24..) F. Prop *334-22. F: K E FMconnex.) pT'GKic = s'GI"K [*334-21. *330-52] *334-23. F K E FM connex.). coflX"K = 'G"'c [*'334-21. *331-4] *334-24. F: K c FM connex.). 8'Ka3 Cconnex Dem. F. *334-21. *331-4.)D F:. Hp. X,y 6s%'["K X+Y.x.: (HR):R E6Ka:xRy. v. y-lx:.)DF.Prop *334-25. F K e FM connex. C"K = U"K [*334'21. *330-52] *334-251. F: K C FM. K, = K V CnV"K.:).pP"cK, = S'U"K Dent. F. *40-18. *33-22.:) F: Hp.). K = PW'GK (1) F. (1).*330-52.)DF. Prop *334-252. F K C FM colx. 8'Kae connex.). K, K v Cnv""K Dem. F.*41-11.) F: Hp. L E K,. x = L'y.:). (g-R).1 C 6K V Cnv"K..xly. [*331-42-24]. LECK v Cnlv"K (1) F. (1).*330-6.*331-12.:) F. Prop *334-253. F: K E FM conx. C'K, = "K.) K C FM connex Dem. F. *330-52.)F: Hp. a. P'C"K = S'P'fK. [*331-1]). 2[! p'G'%, ACOUX'K: F. Prop *334-26. F: K C FM conx): K C FM connex. =. C'a connlex. -. K, = K vCnvl"K.=-. C"K,c= GI"K [*334-21-24-2-25V2-52253] *334-27. F. FM connex = FM A ^K (A'P"K = COnlX'K. K 4 t'A) Dern. F.*331-1.:) F: K C FM. K + t'. A"'C= C~flX'K..k C connex. [*334-26. (*331-02)]).KC FM con nex (1) F.*334-23. (*334-03))F: K C FM COnnex.. S'P'K=- conlX"K. KC + t'A (2 F(1).(2).) F. Prop 25-2

http://quod.lib.umich.edu/cgi/t/text/text-idx?c...stmath;rgn=main;view=text;idno=AAT3201.0003.001 (194 of 246) [5/26/2008 7:31:51 PM]

388 QUANTITY [PART VI *334-3. F:Ke FM Sr.:).s'KaSer Dem. F. *334O09. DF: Hp. D. %c~aC J (1) F. *334-14. DF: Hp. D. 'Kac trans (2) F.*334-24. DF: Hp.)..'Ka cconnex (13) F. (1). (2). (3).:) F-. Prop *334-31. I-:.KceFM. IrS'UI"KEK.:):Kce.FMsr.=E.s'KaESer-t'A Dem. F.*4111.) f-:. HP. EK y.(2 SerR -K x (R u -), [*331 1 1])S'P'"K = COnXK (1) F.-(1).*334-1426.)F: Hp (1):). KeFM trs-Kc FM connex (2) FI-.(2). *334-3.*331P12.) IF.Prop *334-32. F. FM sr C FMap [*334'16&21. *333'101] *334-4. F: KeFM. D"K=P'CK.)K VCnV"K-FM Dem. F.*33-221.:)FI-Hp.).D v VCnv"'K) = G"KV CnV"~C) = "CK (1) F.*330O561.:)F:.Hp.:):1, SCK.:).RIS=SIIR (2) F. (1).(2). *330-52.) F. Prop *334'41. F K 6 FM COnX. D" = PIK.).,cV Cn1v"K E FM conx [*334-4. *331-11] *334-42. F K c FM conx D. K= PCCK Ke i FM connex Dem. F. *37-323. F: Hp.) R, SEc K.Q(S) = GS [*330-4]):G Kg =ccK(1 F. (1). *334-26.)F. Prop *334-43. F:K,6eFMCOIInxnFMtrs.D""K=(IWK K),cFM sr [*334-42. (*334-04)] *334-44. F: KeFMconx.D"'K=-P'K.LC K,.) D'LU='CLG='L=S'UCK Dem. F. *37-323. Z)F: Hp.1R, SEK.).U'(RJIS) =U'S:) F.-Prop

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SECTION B] SERIAL FAMILIES 389 *334-45. F:KeFMconx.D"c= GI"Kc. L, ME K,..'(L IM) = 0"K [*334-44] *334-451. F:Hp *334-44. S EPot'L.:). D'S = U'S =C'S = s'U'Gc [*334-44] *334-46. F:.Hp *334-44. M, NEK,.):f[!LjMAN.=.LjM=N [*334'45. *331-45] *334-5. F: KICEFM COflX n FM asym. (h'Kca) C J Dem. F. (1). Transp.) F:.Hp.) 1?:, Se c le. D. e,(B S C I). [*331-33-23] D. R IS C:. DF. Prop

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*335. INITIAL FAMILIES. Summary of *335. A family of vectors may or may not have a point in its field which is a starting-point but not an end-point of non-zero vectors. For example, the family of which a member is (+, X) C'H', where X e C'H', has such a point in its field, namely Oq; but the family of which a member is (+, X) C'CH, where X e C'H', has no such point in its field, and no more has the family of which a member is (+g X) t C'Hg, where X e C'H'. If such a point exists, it is a member of saI'ck but not of s'Dc"a. Such a point, if it is also a connected point, must be unique, i.e. we have *335-12. F: K e FM.. conx'K - s'D"a e 0 v 1 When conx'c -s'D"K' exists, we call its only member "the initial point of Kc," putting *335'01. init'K = t'(conxc - s'D"Ca) Df If the initial point of K exists, we call K an " initial " family; thus we put *335 02. FM init = FM n (Pinit Df An initial family is asymmetrical (*335'16) and transitive (*335'18), and forms a group (*335-17); and if its initial point is a member of p'C"K,, it is a serial family (*335-3). *335-01. init'K = t'(conx'K- s'D"a) Df *335'02. FM init = FM n (Pinit Df — ~ 4 --- *335-11.: K e FM. a e conx' - s'D-"Ka.. S'"K = SiK'a. t'a - "a Dem. F. *41-43. 33-4. ~: Hp.. 'Ka'a = A (1) ~. *331'23-22. D: Hp. D. S ' Ka'a v= "aa a (2) 4 - F.33112322. 22: FHp.). s'(I"K = S"a u 'K' (3) F. (). (2). (3).) F. Prop

SECTION B] INITIAL FAMILIES 391 *335-12. F:KEFM.:).conx'Kc-s'D",c~ae0vl Dem. -. *335-11.) F iHp. a, be EcoxlXK - sl'D"Klca.). b E &ic'a. [*32-182]). a e S'K'b. [*333511]. a =b:D)F. Prop *335-13. F:. K E FM.): E! init',c. conx',c - s"c [*335-12. (*335-01)] *335-14. F: Ic FMiflit.. KEFM. [! conx'K-s'D,"Kc [*335-13. (*335-02)] *335-15. F:1' FM init.. s'E1U',c = &'K'inlit'K [*335-11.(*335-01)] *335-16. F.FM init C FM asym Demn. F *335-14.DF:.Ke FM init.: (2Ja):a 6s'(1'K:REK-aeD'R.)R.Re LU'1 (1) F*330-52. DF:,c Fil. ae S'"Wc. R e KcACnv"6c. D. a cD'R (2) F.(1). (2).) F:. K e FM init.): 1? E K n Clv"K. DR 1? Re R1'I: [(*334-05)]): K E FM asym:.) F. Prop *335-17. F:KeFMinit.).S'KKCj'K=K Dem. F.- *335-15. D F:. Hp. D R, S E K. D. (3[T). Te6 K.- R'S'init'K =T'init'K. [*331-24-33-32]).(2 T).T EK.R IS = T F.*13-19522.)F:HpI).K6 KK(" F - *331-22 - D F: Hp - D. IC CS6K 6 "IC(2) F. (1).(2). F. Prop *335-18. F. FM init C FM trs Dem. F. *335-17. F:. KeFM init D:1?, Se Ka.)R I SEK (1) F.*334-5.*335-16. DF:.KeFMinit.):R,SEKa.D.RjSGJ (2, F.(1). (2). *330-5 51. DF:. K cFM init.: R, SeKIc3.).-RjISeKIc (3) F.(3).*334-13.)DF.Prop *335-19. F:. K EFM miit.):K EFM connex.=..rnit'K eP'C"KC, [*334-23. (*334-03. *335-02-01)] *335-21. F:K eFMinit.). "aE trans. (h~lK)2 -j J[*335-1816.*334-14-5] *335-22. F:. Kc FMinit.): SK e connex.*=.C 'K= tI"KC.=. ifit'K epWC"K, [*334-26. *335-19]

392 QUANTITY [PART VI *335-23. FI-K.eFMinittr'FMconnex.Le~. Dern. F.*335-19.) F:. Hp.)initce D'IL. v.init'Kce I'L (1) F. *334-21.:) F: Hp.)D. Le K a v Cnv"'Ka (2) F.- *335-11. D F: Hp.:): LE lea.:).init'/c,.' eD'L: Le~ Cnv"Kca.D 1fit'K' C —, U'"L (3) F. (1). (4). *5-17. DF. Prop *335-24. F:.,ceFMinitnFMconnex.1?,ScK. BtS.:): R'rinit'c C D'S. Slinit'K' -,, D'R Dem. F. *71-162.) F:. Hp.):R'init'K C D'S. E. init',c C GI(S IR). [*31331.*315-23].init'Kic,e D'(SI 1?). [*71-162].S'init',c -c D'R:.:) F. Prop *335-25. F::.K~eFMinit.):: ~%caeconnex.=: RSexK.).R,S: D'BC D'S.v. DIS C D'R c,8 ~3C D "Ic.)ap C,3. v. /3 C a Dem. F.*202-135.)F:: Hp. ~Ka e connex.):. S'K Cconnex:. [*211-6.*330-542]:):. R,SEKc.): D'R C DS. v.D'SC D'R (1) F.*71-162.) F:Hp.1R'init'Kce D'S.).init'KEU'(J? S) (2) F.(2).(3).)F:.Hp.R, SCK:D'1?CD'S.v.D'SCI)R::). init'KceG'(R I5) (4) F. (4). *330-4. F Hp H:. 1?, SEcK.:) R,S: D'R C DIS. v. D'S C D'R:.).[*335-22] k.c c connex (5) F. (1).(5).*37-63.)F. Prop *335-26. F:KeFMinitriFMCOnnex.:).Dr K C1 -)I1 Dem. F.*335-24.) F:Hp.R, SEKi.RB[S. R'initb/ e D'S.).S'init'K,c 'e D'R. [*33-43]:).D'B#+D'fS (2) F.(1). (2).) F:Hp.RB, SCe.RB+ S.).D'R =[D'S:)F.Prop *335-3. F: ic FM. initc ep'C"ilc,.)."'ae~Ser [*335-21-22]

*336. THE SERIES OF VECTORS. Summary of *336. In this number we consider a relation between members of K or of I, which, with suitable limitations as to the nature of the family, may be identified with the relation of greater and less. If there is a member of K which takes us from a point z to a point y, i.e. if y ('Kca) z, we say that z is an earlier point than y; thus we regard s'Ka as the relation of later to earlier. If now M and N are two members of K,, and if, for some x, M'x is later than N'x, we shall say that M is "greater" than N with respect to K. This relation we denote by VK, where " V" is intended to suggest that the relation holds between vectors. The definition is: *336 01. V, = MN [M, N e K,: ([]).(M'x) (S'Ka) (N'x)} Df For the same relation when confined to members of c, we use the notation U,; thus we put *336-011. U,=V, e Df In dealing with V, and U, it is desirable to be able to express M'x as a function of M. We wish to consider (say) a fixed origin a, and the various points R'a, Sa, T'a,... to which the various vectors which are members of K carry us from a. For this purpose we put R'a = Aa'R, where "A " stands for "argument," and "Aa'R" may be read "the value, for the argument a, of R." The definition is A, =aR (xRa) Df, whence we obtain *336-101.: E! R'a.. R'a = AaR Then the points R'a, S'a, T'a,..., where R, S, T,... are the various members of /, form the class Aatc", which is thus the same class as &'Isa. The relation A a K correlates the point R'a with the vector R. The vector R is analogous to the coordinate of R'a when a is the origin; thus Aa, K is analogous to the relation of a point to its coordinate. A relation which is more exactly that of a point to its coordinate will be explained in Section C, where, in

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394 QUANTITY [PART VI addition to the above correlator Aa, K, we shall also correlate a vector with its numerical measure in terms of an assigned unit. If ic is a connected family, and a is any point of its field, Aa r K, is a oneone relation (*336'2). If K is an initial family, and a is its initial point, Aa, K is a correlator of s'"I' and K (*336'21), so that in an initial family the class of vectors is similar to the field (*336'22). If K is a connected family, and a is any point of the field, and X is those members L of K, for which L'a exists, then Aa r x correlates the field with X, so that X is similar to the field (*336-24). By the definition of A,, if M e K, and M'a exists, we have M'a = AaM = Aa r K,'M. Hence by the definition of V1, F: MV N. -. (a). (Aa K,'M) ('Kca) (Aa r K,'N). - (aa). M(, 1 Aa;S'Ka) N, by *150'41. Similarly F: P U, Q. -. (a). P (K A a;'cKa) Q. Now in a connected family, if a and b are any two members of the field, and P, QC K, (P'a) ('Ka) (Q'a). -. (P'b) (K') (Q'b) (*336-38); hence K 1 Aa;S'ca = Ic Ab;'Ka, and hence UK = AI a;s'Ka (*336-43). Since K 1 Aa is one-one (by *336'2), the above gives an ordinal correlation of U, with (S'K) Aa"Kc (*336'461), i.e. UK is ordinally similar to s'Kc with its field confined to those points which can be reached from a by vectors which are members of K. If K is an initial family, it follows that UK is similar to S'Ka (*336'44); if not, U, is in general only similar to a segment of S'Ka (in the sense of *213). It should be observed that, 1 Aa'x is the member of K, which takes us from a to x, and K 1 Aa' (if it exists) is the member of K which takes us from a to x. Thus K1 Aa;'Ka is the series of vectors which take us from a to all the various points which can be reached from a by members of K, the order of the series being that of the points to which the various vectors take us from a. If K is a connected family, U, is the relation which holds between two members of Kc

when one of them is the relative product of the other and a third (other than the zero vector), i.e. *336-41. F: e FM conx. D. U, PQ [P, Q: (T). Te. P =T IQ

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SECTION B] THE SERIES OF VECTORS 395 This is for many purposes the most convenient formula for UK. If, in addition, we have D"K = ("cK, a similar formula holds for VK, i.e. *336 54. F: ic eFM conx. D"Kc = (". D. = MN2 M, Ne IC,: (T). T a. M= TI N If Ec e FM conx, V, is contained in diversity (*336'6); if c is also transitive, VK is transitive (336'61); and if K has connexity, so has V, (*336'62). Hence if K is a serial family, VK and UK are serial (*336'63'64). In addition to the above-mentioned propositions, the following propositions in this number are important: *336-411. F:. FMconx. s'c "KC K.: PUKQ.R e K. D.(P R) UK(Q i R) *336-511. F:. K e FM sr. v e NC ind - t'0. D: R UK S.. Rv UK S *336-53. F:. K e FMconx. M, Ne K,.): MVN.. NV,M The present number is important, since VK and UK are the general relations from which greater and less are derived, and the subject of magnitude is therefore intimately dependent upon them. *336-01. V, = MN {M, N e,: (3Jx). (M'x) (.s'K) (N'x)} Df *336'011. UK= VK C K Df *336'02. A, = R (xRa) Df *336-1. F: xAaR. =. xRa [(*336-02)] *336-101.: E! R'a. D. R'a = A,'R [*336-1] *336-11. F: (A rK) R.. R e a. Ra [336-1] *336-12. F. K"a = Aa"K = D'(Aa c K) Dem. F. *41 11. F.'a = -^ {([R). R e cK. Ra [*336'1] = {(sR). Re K.xAaR}. D. Prop *336-13. F. D'Aa] K C s'D"Kc Dem. F. *336-12. *33-15. D F. D'A, C D". F. Prop *336'14. F: C 1 -- Cls. D. Aa, r c 1 — Cls Dem. F.*33611.)F: x(Aa, c)R.y(Aa K)R..Re. xRa. yRa (1) F.(1). 71'17. D: Hp. Hp(1).. x=y (2) F. (2). *71'17. F. Prop

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396 QUANTITY [PART VI *336-15. F:/KCcr'a.aea.:).WI(A4,r/)=K' Dem. F. *33611I.:)F:ReU'((AarK).=.(aX).REic.x~a(1 F. (1).-(*330-01))1F. Prop *336-16. 1- a ecornx',c..a es'I',c. Aa"(C v Cnv"c) =sCK Dem. F.*3311..*336-12.) F- a E conx',c. =-. a e s8'P'xi. A,,"Ic v Aa"'CIIv =c S'G"(IK (1) F-. (1).*37 22.)D F. Prop *336'17. F-: ce FM conx n FMtrs - 1. P = s'Ka. A).A/"=P*'Ca Dem. F *334-14-18.) F:Hp.D. P*'a =P'a vI rsl'CU"'c'a [*331-22-23] =htl [*336-12] = A a"KI: D F. Prop *336-2. F: i eFM conx. a cs'P'ic1. D. Aa 4E 1 -+ 1 Dem. F. *336-14.)D F: Hp.). Aa r K, El I- CIS (1) F.*33611l.)DF:Hp. x(AarI,c) L. x(Aa4rKc) M.)L, M cKL, a. xMa. [*331-42] D. L =M (2) F. 1).(2) DF. Prop *336-21. F:KeFM.a=init'c.:).AarKe(s'UI",) gYiiK Dem. F.*336-2.)F:Hp.D. Aa r KEl —+I (1) F.-*335-15. *336-12. D)F: Hp. D). D'Aa rK/c= S'CL"K (2) F.*336-15'. D F:Hp.:). G'Aa rIC = (3) F.(1).(2). (3). DF. Prop *336-22. F K E FMinit.).(s'P'K/) smiK [*336-21] *336-23. F xe FM conx. aesl'P'K.X K, AL(a cU'L). D Aa r Xe6(s'OI"K/) gi X Dem. [Hp] IDFH.DDI(~x=5 {x"(HL).Lex~.xLaI [*331-4] = 8'P'Kc/ (2) F. *336-11.:) F:Hp.). (A~ a X) = L t (a1x). L E X. xAa} [Hp] =X(3) F.(1).(2). (3). DF.Prop

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SECTION B] SECTION 13] THE SERIES OF VECTORS39 397 *336-24. F:Hp *336-23.D). s)M X [*336-23] *336'25. F:KcFMconx.a,b S'UF"K.X =K, rL (aeUG'IL). L=AM (bePIM).).-Xsm~t [*336-24] *336-26. F:'ceFM.aeconx',c.X=,cvCnvl"1?(REK-caaED'R).).D Aa r X e (s'(Itc)0 9-n- X [*336'23. *331-48] *336-3. F:.KC1I-0 CIS.:: R(K1 AaP) S. R, S K. (R'a) P (&a) Dern. F. *115011.)F: R (K 1Aa;P) S. (HX, Y). R, SEK. xAaR.yAaS.XPY. [*336-1] - (2x, y).R, SeKc. xRa.ySa.xPy (1) F. (1).*71-36.) F. Prop *336-31. F:K E FM conx *a Es'U["Kl.).KaC D'(#c 1A,,&'Kc) Dem. F. *336-3.) F.*331P22.) F:Hlp. RE 1%.). REKaj. Irs'8'P E/c. Ra= R'(I[" s~'Gi'%c)'la. (1)] D. Re D'Qc 1lAa;h'Ka): D F.- Prop *336-311. F: K cFMconx -1. a ES'GT"K.). I SCI' s'G I",eU'(Kl Aa~'Ka) Dem. F.-*336-3. F:.Hp.:):SePI(K1A,;%~ca) =(aR,T).R,S~EK. TElca.R'a=T'S'a: [*331P22]:): I rsWI",cK eP1(Ic 1 Aa,;h'K). (HR, T) 1? R K.- T iKa. R'a = T'a. [*330'52].a! Ka (1) F (1).-*334-18. D)F. Prop *336-312. F: c FM conx -1.). C'Qc 1Aa;~'Ka) =I K[*336-31-311] *336-313. F:KE FM coux rnFMasym. a eS'P',K.). D'(Kl1Aa;h'ia)=Ka Dem. F.- 336 3.)D F:. Hp.) IrS'c"'KceD'(K1A,,;s'Ica).= (aS,T).SeK.Teica.a= T'IS'a (1) F. (1).*334-5.):Ip) ~'~c. ',I~K)(2) F.(2).*336-31)DF. Prop

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398 QUANTITY [PART VI *336-32. F:KcEFM.aeconx'K.X,=KAR(acD'R).:). C'(1 Cnv'%K) 1 A,,;'KaJ = K v Cnv"X Dem. F-. *336-16. *334-18.:) I-: Hp.).C'YKa = W'(K v Cnv"Kl/) 1 A a. [*150-23]).C'jQK v Cnv"'K) 1 Aa4'Ka' D'(K V Cnv"'K) 1A'a [*336-15,11] = K VR B (ax). B c Cnv"ic. xRa} [Hp] = K v Ctv"X:) F-. Prop *336-34. -: K eFM. a = iit',K.. (Kl1Aa~'Ka) smor (h'Ka) Dem. F. *336-21.) F: Hp.-). K 1Aa61 -+ 1.U'1Kl A a= O"Kb: F. Prop *336-35. F: K c FM. aC CcOhX'K.. t(K v CnvI"K) 1jAa~k'Kla}smor (kKa) [*.336-216] *336-351. F: K c FM conx. aE 'C 8c"K. (K 1j'a;~'Ka) smyor (~'Ka) ~ Aa"K Dern. F.*150-37.)F: Hp.)./K1Aa;s'Ka=K1Aa; (s'Ka) Aa AC~K (2) F.(1).(2).) F.Prop *336-36. F:.KeFMconx.LMEK,.a, bEUI'LAU'"M.TcK.:): L'la =T'M'la.. L'b = TM'b: L'a= T'M'a.-. L'b =T'M'b Dem. F. *13-12. D F:. Hp.- NE KL a =N'b. D L'a- T'M'a..L'N'b = T'M'N'b. [*330-63] NI'L'lb = N'T'M'b. [*71P56].L'b = T'Mb (1) F.(1).*331-4.DF:. Hp.)L'"a = TM'la.EL'lb = T'Mb (2) F. *71-36 2.)F:. Hp.:L'"a = TM'a E.Mla =T'L'a. [(2) ML;M~b9&iT. M'b = [*71V362] EL'~b = TM'b (3) F.(2). (3).) F.Prop *336-37. F:.KEFMconx.L,MEK. a, b eP(LrhP(M.: Dem. F. *336 36.) F:. Hp.): (aT).TE6Ka. L'a =T'M'la. =. (2T). TEKa. L'b= T'M'b:.DF. Prop

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SECTION B] THE SERIES OF VECTORS 399 *336-371. F:. KeFCffox.L,MEm,. a r (Ln(Pt LIK& M..(')@c)Ma)[*336-37. (*336-01)] *336-38. F:. KeFMCOnx.P, QE6K. a, bES'G""K): *336-4. F: K e FM con x * a E6 SWJ"K.) UK P Q {P, Q E K. (P'a) (9'Ka) (Q'a)} Dem. F. *336-38.) F:.- Hp.)D: be Es'P'K. (P'b) (9'Ka) (Q'lb). b E s'EII"K. (P'la) (9'Ka) (Q'a): F (1).(*33 6 011).) F.Prop *336-41. F:KeFMconx.:). UK=PQ{P,QEK:(3T).TEKS.P=T)QI Dem. F.*4111I.:)F:Hp. a es'T"Kl. (P'a) (~'Ka)

University of Michigan Historical Math Collection: Principia mathematica, by Alfred North Whitehead ... and Bertrand Russell.

(Q'la).:). (HT). T eKa P'a= TQ'"a. [*331 32-3324].(T).T eKa.P=TIQ (2) F. (1).(2).- *336,4.)F. Prop *336-411. F:.KEFMconx.s'K,<"KCK.:):PUKQ.REK.:).(PJR)UK(Q IR) [*336-41] *336-412. F:Hp *336411.P, Q, R EK.(P IR)UK(Q IB))PUKQ Dem. F. *336-41)F: Hp.:). (2T). T Ka. P R= TjQ IR. [*330-51)(aT). TE Ka R R IP=R IR IT IQ. [*336-41] D. PUKQ:) F. Prop *336-413. F:.Hp*336-411.P,Q,REK.:):PUKQ.E. (P JR) UK(QIIJR) [*336-411-412] A *336-42. F:K EFM conx *a ep'D"K.. 1&, LMA IL, ME K,. (L'la)(k'Ka)(M'la)} Dem. F. *330P54.:) F:. Hp. L, ME K,. D: a E ALnGM F (1).(*336-01). DF. Prop *336-43. F:KEFM conx. a eS'U"'K. UK K1Aa;k'Ka Dem. F. *336-4101.:) F: Hp.:). UK = PQ {P, Q E K. (A,'lP) (h'Ka) (A,,'Q)} [*35-7] =PQ (A a r K'P) (~'Ka) (Aar~ K'Q)} [*150-41.*336-2] =K 1 A a'S'Ka: F. Prop

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400 QUANTITY [PART VI *336-44. I-: K E FM init. 5. UK, smor (~i'Ka) Dern. F. *336-41. 3F:Hp1. a= init'/. D. U,,K=1 Aa~h'K'a (1) F. *336-21.)DF: Hp. a =init'K.).K1AaE I-+I.U'f(K1JAa)SUK (2) F. (1).(2). *334-19.) F. Prop A *336A45. F: KEFM -acconx'K.X=KA R We D'IR.) VK ~ (Kc v Cnv,"X) =(Kc v Cnv"ccK) 1JA,;~, Ka Dem. F. *41'11. (*336-01).)D F:. PtVY~(Kcv Cnv,"'X)1Q.-: P, QeK v Cnv"'X: (ax, T).TE Ka. Px= T'Q'x (1) F. (1). *336-36. D F:: Hp.)3:. P {VYK w K Cnv "X)} Q. P, Q c K u Cnv "X: (at T). TE Ka. P'a = T'Q'a: [*14-2l.Hp] ~P, QE K vCIv "K (a T).T eKa. Pa-=T' Q'a: [*41-11] P, Q E K v Cnv C"K. (P'Ca) (~'Ka)(Q'a): [*336-3] P {(K v Cnv"tK) 1 Aca;~'Ka} Q:: D F. Prop *336-46. F:Hp *3.36-45.).,(K vCnv"'X) smor (g'Ka) [*.336-45-216] *336-461. F: K e FM conx. a E s"K "I U.U smor (~'Ka) ~ (Aa"'K) [*336-351-43] *336-462. F: K E FM coflx A FM trs.- a e s'GI'cK.- P = ~"K. *U, smor (P ~ P*'a) [*336-461-17. *334-17] *336-47. F: Kce FM conx.:).,ca C D'U,, [*336'31P43] *336-471. F: K E FM conx -1I.:). K = C' UK [*.336-312-43] *336-472. F: ce FM conx A Fffasym.:). Ka = D"UK [*336-313-43] *336-51. F:. K FM sr.R, SEK. veNC ind -t'0.: Dem. F. *333-42. *334-32. *330-57. *331-42. F:. Hp.:): TE Ka. -R'a = TISI'a.. K'va= TvIS&'a. F. (1). *41 11. D F:.Hp.)D: (B'a) @h'Ka) (S'a).). (Th'la) (~'Ka) (Svga) (2) S, R F. (2) A S. F:.Hp.:): (S'a) (~I'Ka) (R'a).) (Sv~ja) (.I'ca) (1?vla) (3) F. (3). (4). *3.34-3.) F:. Hp.) (R'a) (&'Ka) (S'la).) f~ (Rv'a) (,~'Ka) (Sv'a)} (5) F. (2). (5). D F. Prop

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SECTION B] THE SERIES OF VECTORS 401 *336-511. F:. K eFM sr *v ENCind -t0. RUES.. RvU~Sv [*336-51A4] *336-52. F:. KEFM conx.Q,1R,STEK.xeQ'(QJR) rd1'"(SIT).): (Q IB) Vc (S IT)..(&R'x) (A'ica) (Q,'T'x) Dem. -. *336-371. F:. Hp.:) QIR c( ) 2.P). PeIa. Q'R'x= P'S'T'aJ(F.- *330-56.) F:. Hp. Pe Ka.):Q'R'x = P'S1T'x. =. Q'R'x = S'1P'T'x. [*71-362] =. Rix = Q'CS'(P'T'fx. [*330-54-56] =. R'x = S'Q'P'T'lx. [*71-362.*330-5] =. S',Rx= P'QT'x (2) [*41-11] 2.(S'R'x) (9"'a) (Q' T'x):.)D F. Prop *336-53. F:.KEFMCODX.MNeK,.:):MV,,N.=2.NJ&M Dem. F. *330'5-54. F:Hp. Q,R,S, TeK.M==QI R.N=SI T.ae6sU"",K.x=Q'R'S'T'a.:). E! M'x. E! N'x.E! M'x. E! Nx (1) F.- (1). *336-52.:) F:. Hp (1).):M VKN..(S'R'x) (h'tca) (Q'T'x). [*336'52] ( )Tc( ) [Hp] 2NVI&M (2) F.(2). *331-12.) F. Prop *336'54. F: K E FM

COnX. D =KG"K V = MN{M, N eK,:(aT). TE6Ka.M Tj NJ Dern. F. *334-46)F:. Hp. M, NE K,. (aT,x).TEKa.M'lx=T'N'x.=E.(HT).TEKca.M=TIN (1) F(1). (*336-01). -) F. Prop *336-6. F ce FM conx,.V, C J Dem. F. *331-23.)D F:. Hp.: M V5,N). (ax). M'1xt N'1x:.) F. Prop Observe that, by the conventions explained in *14, " M'x Az N1x " implies E!M'x. E!N'x. From "(ax). -..(M'x = Nx)" we cannot infer MtN. R. & W. III. 26

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402 QUANTITY [PART VI *336-61. F:,ceFMconx trs.). ~ trs Dem. F.*330-612. F: H~p.L, M, Ne c. D. 2!CU'L ^PIM nGIN(1 F. *336-371.)F:Hp.L 1&,M. M VCN. ac G IL A PIM A GIN.) [*3:)34-14]).(L'a) (h'K) (N'a). [(*336-01)])LVKN (2) F.(I).(2). DIF. Prop *336-62. F x Ec FM3 connex. D. ~~ connex Dem. F. *330-61.) F: Hp. L,M ME K,.:).~! PL ALPM (1)M F. *334-24.)D F:. Hp. L, MEK,. a c PL AN PIM.): L'a = M'a. v. (L'a) (hxa) (M',a). v.(M'la) (1'/a) (L'a): [*331A42.(*336 01)]):L L= M. v. LI&~M. v. MVKL (2) F.(1).(2).) F.Prop *336-63. F: KeFM sr.). 1& ~Ser [*336-6-6162] *336-64. F: KcEFM sr.)U. ~ Ser [*336-63]

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*337. MULTIPLES AND SUB-MULTIPLES OF VECTORS. Summary of *337. In this number, we are concerned with the axiom of Archimedes and the axiom of divisibility. If K is a family of vectors, K obeys the axiom of Archimedes if, given any two points x, a in the field of E, and any vector R which is a member of K, there is some power Rv of R such that Rv'a is later than x. That is, K obeys the axiom of Archimedes if, starting from any given point in the field, a sufficient finite number of repetitions of any given vector will take us beyond any other assigned point. A sufficient hypothesis for this is that K should be serial. and Cnv'&'Ka should be semiDedekindian (cf. *214), i.e. we have *337'13. F:. eFM sr. P= 'Ka. Pe semi Ded. R e Ka. a e C'P.: x C' P... ([v). v e NC ind - '0. xP (RY'a) The hypothesis P = 9'ca, which appears in the above proposition, is often notationally convenient. It will be observed that s'Ka gives us the series in the opposite order to that in which it is usually wanted; hence the introduction of the above relation P tends to avoid confusions. A family K is said to obey the axiom of divisibility when, given any member R of Kc, and any inductive cardinal v other than 0, there is a member L of K such that LP=R. When this axiom holds, every vector can be divided into any assigned finite number of equal parts. We shall in the next Section (*351) define a family for which this holds as a "sub-multipliable family," denoted by " FM subm." For the present we are concerned to find a hypothesis as to s'Ka from which this property can be deduced. The hypothesis in question is that Cnv'9'Ka is serial, compact, and semiDedekindian; i.e. we have *337-27. F:. K e FM sr. Cnv''Ka e comp n semi Ded.): S e K. I e NC ind - tO.). (L). L e K. S = L The proof proceeds by taking two points a, x in the field of K, of which a is earlier than x, and considering the class XK= Ka R (R"'a Px}, 26-2

4~04(QUANTITY [PART VI i.e. the class of vectors such that v repetitions of them, starting from a, do not take us as far as x. It is easy to show that, when P is compact, this class has no maximum (*337-23), and therefore, when P is also semi-Dedekindian, has a limit, whose vth power is the vector which takes us from a to x (*337-26). Hence our result follows. 3337-1. F: K e FM. P8h'Ka. R e lea. a e 'P.). R*'a CP"CR*'a Dem. F.c*90-16. *41-141.): Hp. xR~a. y =R'I. ' ye R*Ca. xPy. -4~ [*37-1]. xe P"R'*a: F.- Prop *337-11. 1-: K e FM connex asym. P = h'Ka. RE Ka. aeC'P..seqp'1R~a = A Dem. F. *206-1.) 1: Hp.. seqpCR*'a =p'P"IR'a - PC p'P"CR*'a (1) F.*330-542. *40-61)F: Hp. xe p'P"R'a.).xe D'R. [Hp]).(ac). x = R'c. cPx (2) 1-.*90-172.)F: C e R*'ca. R'ce R*'a (3) F. (3). Transp. *200-5. *334-a5. DI-: Hp (2). x R'c. D. c'- e R*'a (4) F.*37-1. D F: ce P"*C'a..(b).bE R*'a cPb (5) F. (5). *2082.D F: Hp. ce P"1?*'a. x = R'c. D. (:b). b e R*'a. xP (R'b). [*90-172] Xe P'"R*C'a (6) F.(6). Transp. *20053. -: Hp (2).x = R'c.). cr%,e P"R*'a (7) F.(4).(7). *202502. *334-24. D F Hp (2). = R'c.). c ep'P"R*I'a (8) 'S 4- -—), F. (2). (8).)F: Hp (2).). x e P11p'P"PIR*'a (9) F.1.(9). DF. Prop *337-12. F: KEFM sr.P = h'Ka.PEsemi Ded.R E Ka.aeC'P.).P"IR*'a=C'P Dem. F. *3371.)F:Hp..E a maxp'R*'a. [*205-7]). j maxp'P"9R*a (1) F.(1). *206-33. *337 11.) F: Hp..D! seqp'P"R*'a (2), F.(1).(2). *214-7. F. Prop *337-13. F:. Ke FM sr. P = hca.PesemiDed.Re Ka. a e CPs. D: xGE 'P.).(av) vNC id '0. "OxI (Rva) [*33712. *301-26]

SECTION B] MULTIPLES AND SUB-MULTIPLES OF VECTORS40 405 *337-14. F: IEFM sr.P=s'Ka.Pesemi Ded.).U,,esemiDed [*336-462. *214'74'75] *337-2. F: KeFMcoflx.LUICR. R#+Itrs'Ulifx. D. LU, (R IL) Dem. F. *336-41. D F: Hp. D. (ST). L, B e K. T e "a. L = Tj B. [*330-31])(31T). TeKa.R IL =T.L =T IR. [*13.195].R IL eKa. L =(RJIL)jIR. [*330'5.*336-41] D. LU(R (B L): D F. Prop *337-21. F: IeFMconxmFMtrs.J6 Ka -v eNC ind - tfO- t'1l.).BR",U,, Dem. F.*334-162.*301-23.:) F:Hp.).Rv=Rv-cl B (1) F. *334-131.:)F:Hp.). R,BRY,,RV,- 1EKa (2) F. (1). (2). *336'41.:) F. Prop *337-22. F:KecFMsr.P=~'Ica.PeCCOMP.a.Px.vENCind-L'0.). (aR). Re Ic. (RV'a) Px Dern. F.*270-11.:)F:Hp.). (y).aPy.yPx. [*41-11])(gR, y). Re Ic. y =R'a. (R'a) Px (1) BR a F.(1)..)F:Hp.Re~a.(BR'a)Px.:).(aS).Se6a.(S'IRv'a) Px (2) a F. *336-64.)F:. Hp(2). SeK&a.(S'R"'la) Px.):R==S. v.RU,CS. v.SU,,R: [*3.36-511-4]:RB= S. v.(By +c 1a) P(S'R~l'a). v.(Sv +c 1a) P(S'B,"'a) (3) F. (1). (4). Induct.) F. Prop A *337'23. F: Hp *337-22 * K=a B R{(R"'a) Px}.).X = h"1X Dem. F.*336 511I.:)F: Hp. Re6X.SUR.). (SI'la)P (B"'la). (Rv"'a)Px. [*334-3.Hp] D. SeX (1 F. *337-22. D F: Hp. B e X. D. (2jS). S e Ica. (S,,'R",a) Px. [*330-57'5. *334-13].(21S). B S e,ca. {(RI? S),"'aJ Px. [*336-41]).(E[S).R B tea. { (R I S)""al4Px. R UK (R 5). F.(1).(2).) F. Prop

406.. QUANTITY,[PART VI *337-24. F:.Hp *337-2~.L =til(UK)'X.)* {,(L,"Ia) Px} Dem. F. *206-2.) F: Hp.. L,.e [HP]). (RLI" a) Px}: F. Prop *337-241. F: Hp *337 '24.)D. { xP (LV'Ia)} Dem. F.*337-223.) F: Hp. Be E.) R L [*332-53-241.*3.34-131]). B L x (B L)v RIJ LY, [Hp]). (Rt"P'a) Px. F. *337-23. D F: Hp. RE 1cc- X. D. tLURl. [*336-511] D. t(R,"'a) P (LV"'a)} [*330'5.Hp.*334-14]). { (R"'ax) P (Lv'fa) J (2) F. (1). (2).:) F: Hp.:). r-(aR). RE Ka. (RB,'x) P (Lv'a). [*3:37-22.Transp]). xP (LI"'a)J:)D F. Prop *337-25. F:Hp*337-24.:). P =/iClAa,'x Dem. F.*337-24-241.:)F:Hp.).L,"a-x::)F.Prop *337-26. F: Hp *337 23. P E semi Ded.). {tl (U,)',P' -=xj Aa'x Dem. F. *337'21.) F:. Hp.)D: REX. DR (B'a) Px: [*336-4] D: KjAa'x6P'U"X (1)t> F.(1).*337-23-14.)DF: Hp.. E! tl (U,)'X (2). F. (2).*337-25.)D F. Prop *337T27. F:.Kxe FMsr. Cniv'ht/x&acomp n semi-Ded.:): SEK.veNCifld-t0.:). (afL).LEK.S=Lv, [*337-26]

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SECTION C. MEASUREMENT. Summary of Section C. In this Section, the "pure" theory of ratios and real numbers developed in Section A is applied to vector-families. A vector-family, if it has suitable properties, may be regarded as a kind of magnitude. In order to derive from the "pure" theory of ratio a theory of measurement having the properties which we should expect, it is necessary to confine ourselves to some one vector-family; that is, instead of considering the general relation X, where X is a ratio, we consider the relation X t Kc, where tc is the vector-family in question; or sometimes we consider X K,, or sometimes X (v C nv"c). Concerning ratios with their fields thus limited, which are what we may call "applied" ratios, we have to prove various propositions. (1) No two members of a family must have two different ratios. This is proved, for an open and connected family, in *350'44. (2) All ratios except Oq and oo q must be one-one relations when limited to a single family. This is proved, for an open and connected family, in *350'5; with the same hypothesis, 0q is one-many (*350*51). (3) The relative product of two applied ratios ought to be equal to the arithmetical product of the corresponding pure ratios with its field limited, i.e. if X, Y are ratios, we ought to have X K I Yt =(Xx8Y) K or X Y C, =(X x8 Y),. That is to say, two-thirds of half a pound of cheese ought to be (2/3 xs 1/2) of a pound of cheese; and similarly in any other case. For any open connected family, we have (*350'6) X,C,, Ye/C, (X X, Y)C, but in order to obtain an equation instead of an inclusion, it is necessary (*351'31) that K should be "submultipliable," i.e. that if R is any member of c, and v any inductive cardinal other than zero, there should be a member of c whose vth power is R. The class of such families is denoted by "FMsubm," and considered in *351.

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408 QUANTITY [PART VI (4) If X, Y are ratios; and T is a member of the family K, we ought to have (X c'T) (Y ic'T) = (X +8 Y) rT, that is, two-thirds of a pound of cheese together with half a pound of cheese ought to be (2/3 +1/2) of a pound

of cheese, and similarly in any other instance. This property is shown, in *351P43, to hold for any open connected submultipliable family in which all powers of members are members. In any open connected family, if R, S, T E c, we have RXT. SYT.).(R IS) (X + Y) T (*350-62). The remainder of the hypothesis of *351*43 is required in order to prove (a) that X KcT, Y cT and (X+, Y) Kc'T exist, (b) that (X t 'T) (Y 'T), which is the R IS of *350'62, is a member of K. As applied to /,, we have to take the representative (cf. *332) of the relative product; if L e K,, we have (*351 42) rep,,'(X C K,'L) I (Y KI,'L)} = (X +8 Y) C K,'L, provided c is open and connected and submultipliable. The fact that the above propositions can be proved for suitable vectorfamilies constitutes the reason for studying such families, as we did in Section B. The proof of the above propositions, together with other elementary properties of applied ratios, occupies the first two numbers of this Section. We proceed next (*352) to consider all the rational multiples of a given vector in a given family, i.e. all the members of a given family K which have, to a given vector T, a ratio which is a member of C'H', or, alternatively, all the members of K, which have to T a ratio which is a member of C'Hg. It will be observed that, in virtue of *307, if R and S have a ratio X which is a member of C'H', R and S have the corresponding negative ratio X Cnv. The members of K which have to T a ratio which is a member of C'H' are those vectors R for which we have (X). X e C'H'. RXT, i.e. using the notation of *336, those for which we have (3gX). X e C'H'. RATX. Thus they constitute the class K n AT"C'H'. Assuming that Te K, the vector which has the ratio X to T is c1 AT'X. This is the vector whose measure is X when T is the unit. Thus K1 AT CGH' is the correlator of a vector with its measure. It is easy to prove (*352'12) that K 1 AT C'H' is one-one.

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SECTION C] MEASUREMENT 409 We can arrange the vectors which are rational multiples of T in a series by correlation with their measures, putting vectors with smaller measures before those with larger measures. The ordering relation is T,, where T,= KAT;H' Df. Similarly the members of K, which are positive or negative rational multiples of T may be ordered by the relation T,,, where T,,=EK, AT;Hg Df. We prove that change of units makes no difference to T,, i.e. if S is any member of K which is a rational multiple of T, then S, = T, (*352'45). The corresponding proposition holds for T,, if S has a positive ratio to T, but if S has a negative ratio, S,, = T,, (*352-56'57). If E is a serial family, T, is the converse of U, (cf. *336) with its field limited to rational multiples of T (*352'72). This proposition connects the generalized form of greater and less represented by U, with the form of greater and less derived from greater and less among the measures of vectors, since it shows that, in a serial family, the vectors which have greater measures come later in the series U,, and those with smaller measures come earlier. We next proceed (*353) to consider "rational" families. These are families in which every member is a rational multiple of some one unit T, i.e. in which (a[T) E Teca. K C AT"C'H'. It is obvious that, given any family, the rational multiples of one of its members constitute a rational sub-family. In a rational family, rationals are sufficient for measurement, and irrationals are not required. If the family has connexity, it will be serial; in fact, if T is one of its vectors and a is a member of its field, we have (cf. *353'3233) v v U,= 1 T AT; H'. 'Sa = Aa; Kc 1 AT;H'. Thus both U, and S'Ka are ordinally similar to H': AT"K. If K is submultipliable, U, is ordinally similar to H' (*353'44). We proceed next (*354) to consider "rational nets," which are important in connection with the introduction of coordinates in geometry. A rational net is obtained from a given family, roughly speaking, by selecting those vectors which are rational multiples of a given vector, and then limiting their fields to the points which can be reached by means of them from a given point. In order to make this more precise, we proceed as follows: Let us define as the "connection" of a with respect to K the class Aa"c,, i.e. all the points which can be reached from a by a member of K,. We will now define as the " a-connected derivative of K" the class of relations obtained by limiting

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410 QUANTITY [PART VI the field of every member of K to the connection of a with respect to K. This class of relations we denote by cxa'c, putting cxa'c = C (Aafc,)"Kc Df. Instead of c, we take, in order to obtain a rational net, all the rational multiples (in K) of a given member T of K, i.e. COT,. Then cx,'C'T is a rational net, namely the rational net associated with the origin a and the unit vector T. In proving propositions concerning the rational net cxa'C'T,, we often require the hypothesis that c is a group. In order to avoid having to make this hypothesis concerning our original family, we construct a closely allied family, which is always a group when K is connected. This family, which we call Kg, is obtained from K by including the converses of those members of c, if any, whose domains are equal to their converse domains, i.e. we put 4 -Kg = K v Cnv"(c n D's'"PK) Df. Then if K is a connected family, Kg is a connected family wliich is a group (*354'14'16), and (Kg), = K, (*354'15). Then putting X = Kg, we take cxa'C'TA rather than cxa'C'T, as the rational net to be considered. If K is an open and connected family, this rational net is a family which is open, connected, rational, transitive and asymmetrical (*354'41). We proceed next (*356) to the application of real numbers to vectorfamilies. For the application of real numbers, it is essential that our family should be serial. Given a serial family in which a given vector S is the limit (in the series U,) of a set of vectors which are rational multiples of another vector R, it is natural to take as the measure of S, with the unit R, the limit of the measures of the vectors whose limit is S. It is convenient to take our real numbers in the relational form given in *314, i.e. if: is a segment of H, we take '8: as the corresponding real number. Thus positive real numbers are the class S"C'O, while positive and negative real numbers together with zero are the class s"C'Wg. If F e C'E, a vector which has to R a ratio which is a member of 4 has a measure which is less than sh'. The class of all such vectors is '4T'R, i.e. if X = s':, it is X'R. The limit of such vectors in the series U,, if it exists, will naturally be taken as the vector whose measure is X. Remembering that U, proceeds from greater to smaller vectors, we see that the first vector which is greater than every member of X'R will be the lower limit of X'R with respect to U,. Hence, if we write X,'R for the vector whose measure with the unit R is X, we

have X,'R = prec (UK)'X'R. Hence we may take as our definition of X, X, = prec (U,) I X K Df. Then X, is an " applied " real number.

SECTION C] MEASUREMENT 411 The properties to be proved concerning applied real numbers almost all require that the family to which they are applied should be serial and submultipliable, and most of them also require that Cnv'h'Ka should be semiDedekindian. Assuming this, we can prove that, if X, Yes "C'O, X,, r is one-one, and, with various hypotheses, (X |K) (Y KC)=(X X, Y)C K (*35631), X, I Y, = (X x, Y), (*356-33), (X,'R) I (Y,'R) = (X +, Y),'R (*356-54). These are the essential properties required of measurement, as in the analogous case of ratios. We might proceed to consider "real" multiples of a given vector, and "real" nets. But these subjects have less importance than in the analogous case of rationals, and are therefore not discussed. The Section ends (*359) with a number on existence-theorems for vectorfamilies. The most important of these are derived from rationals and real numbers. The family whose members are of the form (+ -X) C'H', where X e C'H', is initial, serial, and submultipliable (*359-21). The family whose members are of the form (+-p) C'O', where t e C'O', is initial, serial, and submultipliable, and has Cnv'Y'/ca = O', so that Cnv's'ca e semi Ded (*359'31). Finally we prove that the properties of families are unaffected by the application of correlators, whence it follows that, given any series P whose relation-number is i 1-, or is O' where 0'-+i = 0, there is an initial serial submultipliable family K such that Cnv'SK:a=P. Such a family may be used for the measurement of distances in P. It is of some interest to observe that, given a suitable family K, ratios with their field limited to Ka form a family whose field is Kca. In this family, the zero vector is (1/1) Ka, and the family is connected if K is a rational family. If we wish to obtain a serial family, we must limit ourselves to ratios not less than 1/1, i.e. to ca ' (l/1). This family is serial, and if we call it X, we have (with a suitable hypothesis) h'Xa = U, Ka. It is necessary, however, if we are to obtain a family, that our original family should be submultipliable, since otherwise we do not necessarily have PX Ka= Ka. For this reason, we cannot use the family of ratios without a frequent loss of generality in the resulting theorems. The theory of measurement developed in this Section is only applicable to open families. The application of ratio to cyclic families is more complicated and is considered separately in Section D.

*350. RATIOS OF MEMBERS OF A FAMILY. Summary of *350. In this number we introduce no new definitions, but merely bring together the propositions of *303 on the pure theory of ratio, and the propositions of *333 on powers of vectors in open connected families, especially *333'47'48. We thus find that, if K is an open connected family, and /A, v are inductive cardinals which are not both zero, M {(/) C K.N. -. M, Ne c,.!Mv N. (*350'4). M, Ne K,. rep,'M" = rep.'NA (*350-41), while if R, T are members of c, R (u/v) T. =. Rr = T (*350.43). We prove also, by means of *333'53, that if L and M are members of K, other than I s'(["lc, they cannot have more than one ratio, i.e. *350'44. F: Ke FMapconx.X, YeC'H'.! X K, h Y,.. X = Y We next prove that any ratio other than 0q and 0o q becomes one-one when its field is limited to K, (*350-5), while Oq becomes one-many (3.50-51) and oo becomes many-one (*350'511), Oq being in fact the ratio of the zero vector I s' (I"Sc to any member of C,, and oo q being the converse of Oq. We consider next'the multiplication and addition of ratios, but in this subject we cannot obtain some of the main theorems without the hypothesis that our family is submultipliable (introduced in *351). In the present number, we prove that, if K is an open connected family, and /, v are inductive cardinals other than 0, (t/1) f K I (1/v) t K, C (/Ui) K K, (*350-53), (1/v) K, (s/1) K, = (1/V) K, (*350-54), (,/1): (v (I) / gC,= K {(/ Xo V)/1} K, (*350-55), and (1/,) C K, I (1/V) C [K = 1/(r/ Xc v)J C K, (*350-56). Hence we find that, if X, Y are ratios other than 0 and oo q, X C C, I Y C K, C (X X, Y) K, (*35006), while if R, S, T are members of Kc, RXT. SYT..(R IS)(X +, Y)T (*350-62), and if L, M, N are members of K,, LXN. MYN... rep'(L i M)} (X +8 Y) N (*350-63).

SECTION C] RATIOS OF MEMBERS OF A FAMILY 413 We then prove similar results for subtraction, and thus arrive at the following proposition concerning generalized addition of positive or negative ratios: *350-66. I-: IceFMap conx.L,M, NeK,.X, YEC'Hg.LXN.MYN.). rep,'(L I M) = (X +g Y) C icC2I *350-1. K: KEeFMap. D. IK C Rel num id. K,a C Rel num Dem. F.*333K101.)-: Hp.LCecKa.). Le I-+ 1. Lpo C J (1) F. (1). *300-3.: F: Hp. D. Kta C Rel num (2) I. *333'1-101.:) I: Hp. Le ict -KLa. L c I. [*300-325]). L e Rel num id (3) F-. (2). (3).:) F-. Prop *350-2. F: Hc e FMap conx. abv,a.:. Infin ax Dein. F-.*330-624. *33315. D F:. Hp. L e Kta) A: rfinid'L: [*t121-11-12] D: v e NC induct. D,. (:Ix, y). L (xc y) e v,1 [*120-3] D: Infin ax:. D F. Prop *35021. - F!: FM ap conx -1. D. Infin ax [*33418. *350-2] *350-31. F:./KeFMapconx.,Ct,veNCind-t '.MNE,a.): M (,ulv) N. -=.i Ml' A NP Dern. -. *303-1. (*30202-03). *113-602.) I-:: Hp.):. M(p/v) N.: (a[p, aa,T). p Prn a-. T E NC ind - f0. k = p X * T. V= a-, rT. f! Ma A N1. p t 0.0o- 0: [*333-48] =-:(Hp, a-,). p Prm a- - Te NC id - ff0. p = 0. a+ 0. /I = p xC T v = a XO 7: f! M tA N.L: [*1 13-602.(*302-0203)]: (Hp, a-). (p, a-) Prm (,a, v): f i Ml' A NL-: [*302-36]:t! Ml'A NPL:: DIF. Prop *350-32. 1-:. HP *350-31.:M(/v) N. = rep,'Mv - rep/NIN [*350-31. *333-47] *350-33. F-:.KeFMapconxq.L,veNCind- 'O.M=I 8s'U'G/.Ne n,.): M (lv)N.=-. M=N. =[Y!MvANPL Dem. F. *301P3. *333-2. D F:. Hp.: ae NC ind - t 60. D. MT = M (1) -. (1).*303-1. D F:. Hp.): M(Hv)N...(Hp, a).(p, a) Prm (,, v). f!!MA Ni. [*i333-101]- (ap, a-). (p, a) P~rm (au, v). M~ = N.. [*302-36].M= N. (2) [(1).*331-42] =t! Ml A NIL (3) F. 2). (3). D F. Prop

414 IQUANTITY [PART VI *350-331. H:aceF~fapeonx.L,veNCind-tf0.MeK4,.

N=Irs'U",K.:): M(p.tv)N.=.AI=N.=-.fYMvANIL [*350-33.*303-13] *350-34. H:. KceFMapeonx.vENCind-tf0.MNev,c.): M (O/v)N.E.M =ItrS'P'/c Dem. F.. *303K151.:) H:. Hp.): M (O/v) N. M MC I. aj! C'MAn C'N. [*330'43-61] M M=I r s'P'x:.1 F. Prop *350-35. H:.KeFilapconx.veNCind-t'O.MNefic,.): M(O/v) N. t! MI'A No Dem. F.*301-2.:)H:.Hp.:):f!MvANO.= f[!MvAIrS'(I"K. [*333-101.*331P12] -M= I rS"G'K (1) F.(1).-*350-34.)H F. Prop *350-351. F:. e FMap conx.te NC ind - t'0.): M(k140)N.. N=II ts'PC'fc [*3050-35.4*03-13] *350-4. H:.KveFMapconx.,u, veNCind.r,(p=v=0).): M tYplv),i dN.=. M, NEKx. j! Mv AN~, [*350-3133:331-35-351] *350-41. H:.Hp *350-4.:): M {(,IV) KIc}N.=E.MA,N K,. rep,,'M- repc CNT* Dern. HF. (1). *350-33331P32.) HF. Prop *350-42. F:.Hp*350-4.Q,R,S,TEK.)D: (QIR)(,uI/v)(SI T).=E. Q"lRV = ITh [*350,41.*332-53] *35,043. H:.Hp*350-4.R, TeK.):R(HV/t.)T.-.RvTh [*50.42'r'suK r *350-44. F:/ Fa ox,Y R. XKAY Demn H. *350-4. D H: Hp.:). (aL, M, 1A, v, p, a-) a L, ME K~ L7 MP. t! LI'A MfL. X=,v. Y= p/-. [*333-53]:)qhAx.O=vx.p.X=P1.&v. Y=p/a. [*303-39] D.X= Y:DF.Prop

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SECTION C] SECTION a] RATIOS OF MEMBERS OF A FAMILY45 415 *350-5. F: / EcFMapconx.t, ve NC ind - t0.:). (u/v)-~/c1EI —1 Dem. F -.*350-41.) F-:. Hp.)D: L, M, NE K,.L (pItv) N. M (i/v) N.). repk'J~, = rep,,'Nfl = repKM". [*333-41] D. L=M (1) F.(1). D F:Hp..(11/v) KE -CIS (2) Similarly F: Hp.).(ZIv) KECIS — + 1 (3) F-. (2). (3).)D F. Prop *350-51. F:KceFMapconx.veN`Cind-tfO.:). (0/v) K i, E 1 -+ CIS. (1'(0/i) ~ K, = K, D'(O/v) ~ 1/4 = t"I rs'(J"KI -[*350-34] *350-511. F: Hp *350-51.-). (v/0) ~ ie CIs -+ 1. D'1(z,/) ~ K, = /4. WVOI(/) /=ICI t'lrS'CP', [*350-51. *303-13] *350-52. F:KeFMapconx.XeC'H.:).X~IK,eI-*1 [*35-0-5. *304-34. *333-2] *350-521. F: ceEM apconx.XeC'H'.:). X~ 4eI-+CIs [*350-52-51. *3031I] *350-53.FHp*55..{1) /1(/) (j) Dern. F.*350-4.)F: Hp. L {(i/l) i, M.M {(I/v) ~ cJ N.:). [*333-48] L, M, N Act. fj! L- A M" -. ft! Ng A Mv. x11 [*333-47]).L, M, N e /4. rep,,,Lv, = rep,, 'MP xc=rep,,'NA.. [*350-41]).L {(,u/v) ~ /4}' N: D F. Prop *350-54. F:Hp *350-5.:). (1/v) K~j /4Rgl4) ~ /41I = (/LIi K / Dem. F. *350-41. *332-241.)D (HjM). L, M, N e /4. rep, Ls=,M= rep,,'N*. [*332-22].L, N' it.rp'~ e NL [*350-41].L (ps/v) N:.) F. Prop

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416 QUANTITY [PART VI *350-55. I-: Hp *350-5. {(,it/1 I I} (V/) ~ ~ = {(p, x, v)/1} ~ K4 HF. *350-4.) HF:. Hp.):L [{(I/1)~ K, I} {(V/1) ~,cj] N. = (M[M). L, M, NE tc. ft! L Ai M's. ft! M A: NVv. [*333-47] E.(HM). L,M, NE K,. ftL n MI. M =rep,c"Nv'. [*333-21].LNE K,.ft!L r (replNV)mP. [*3:33-47] ~.L, NE K,. L =repKt"rep,,lNv),m. [*333-24].L, NE K,. L -rep '"(N`)b'. [*350-41.*301P5] ~.L [t (v Xc /A)/1I ~ Kj] N(1 F. (1). *113-27.) F. Prop *350-56. H: Hp *350-O5.:). t(I/,. k) ~ K4J I{(1/,V) ~ K41 = {1/(At X0 vjf ~ K4 = (1/'v) ~ K/4} {(1//p,) ~ K4} [*350-55. *303-13] *350-6. F:e~poxXYCH:.(~,IYK)'X,)K Dem. F. *304-34.:) H*350-54.:) H: K eFM apconx./p, v, p,a e NC induct - tf0.. {(PjV) ~ Ki I {(p/cr) ~ C, = {(1/v),)~ I4 {(Ak/1) ~ K/41I(1/a-) ~ K4 1! {(P/I) ~ K4} [*350'53-54] C' {(1/

V) K~ /1 {(1/L7) ~ ~ ~1 {Qkl/) ~ K/4} {(p/l) ~ K4} [*3505a6'55] C' {1/(V X 0l oj K, I {(Ft X,, p)/l} ~ K [*350-54] C' {(Ot X. p)/(v X., oj} ~ K, [*305-14] C1At/V Xsp/0-1 ~, (2) HF. (1). (2).) HF. Prop *350-61. H:.Ice FM apconx..XeG'H.D: M= (X ~ K,)'N.=M N=-(X ~ K)'M [*350-52] *350'62. H: ce FM apconx.X, YE C'H'. R,8, TE K.RXT.-SYT.) (R S) (X +gY) T Dern. HF. *350-43.:) HF: Hp. X =p/v. Y= p/a.) D. RV = TL. So= [*301-5] R) O. =Taxcff Svxco' - fj'xcp [*330-0-7].(1? S)vxcr- = T (*xco,)+c(vxcp). [*350-41.*306-14] D. (1? j) (X +g Y) T: D H. Prop

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SECTION C] SECTION c]RATIOS OF MEMBERS OF A FAMILY41 417 *350-63. F: KEFMapconx.X, YeC'H.L,M,NeK,. LXN. MYN.). frepk,(L IM)} (X +8 Y) N Demn. F. *350 Al.:) I: Hp.X =Ip/v. Y= p/a-.:)-.rep,,6Lv =repK"N'*. repK'MM = rep8,'Np. [*332-81]:). repK'Lvr = rep, Kv-NIrX. rep,,4Mvxc-O = re t~NvxcP. [*332-33] D. rep/,I(Lv'xc-O MvxcT) = rep,'fN(/xcO')~c(vxP) [*33-28] D. repKI(LI M)vxcO= repc'N(*JxcO) +C frxcP) [*332-82] D. rep.K'{repk'(L IM)}v'xca = rep,,6N(IPxcO) +c (vxcp) [*350-41]) {repK'( ) {~,a)~,(xr, p)}/(v x0 a-)] N. [*:306-14]) {repK'(L IM)} (X +8. Y) N: F F. Prop *350-64. F:Hp *35)0-6.3'.XHY.). {repK'(LIM)} (Y-s X) N Dern. F. *332-13581 F): Hp.)D. repK "LvxcO, Cnv"(repK,'L)vxco' (1) Thence the proof proceeds as in *350-63. *350-65. F:Hp*350-62.D) (II S)(Y-sX)T [*350-64. *308-21] *350-66. F: KEEMapeonIx.L,M,NE'K,.X, YE CHg. LXN.MYN.:). repK/(L IM) = (X +g, Y) ~ icN Dern. F.*350-63.)D F.*350-64.)F:llp(1).XE CH,,. YEGC'H.). W = (X +gY) ~ K,'N (2) F.*350-63.*:307-1. DF:Hp l).X, Ye CH, D.. W= (X~+9Y) ~K,'(N (3) F. *350-34.) F:Hp. X = Oq-).D repK'(L Il) = M [*308&51] = (X +gY) I,'(N (4) Similarly F:Hp. Y= Oq.)D. repK'6(L J) = (X+g Y) 1cN (5) F.(1). (2). (3). (4). (5).:)F~.Prop R. &W. IIL 2 27

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*351. SUBMULTIPLIABLE FAMILIES. Summary of *351. A "submultipliable" family is one in which any vector can be divided into v equal parts (where v is any inductive cardinal other than 0), i.e. in which, if R e /c, there is a vector S which is a member of Kc and is such that S' = R. The definition is *351-01. FM subm = FM ic Re c. veNCind -t'O. R,^.(a2S).Se. R= S} Df In open families, such as we are considering in this Section, S will be unique when R and v are given. But in cyclic families, as we shall show in Section D, there will be v values of S. For example, let K be a family of angles. Then the vector-angle 2jLTr/v has its vth power equal to 27r for any integral value of A, since 2,uTr is the same vector as 27r; and 2pT7r/v has v different values, since, considered as a vector, any angle 0 is identical with 0 + 27r. In the present Section, however, these complications are excluded, owing to the fact that we confine our attention to open families. In virtue of *337*27, a family is submultipliable if it is serial and Cnv's'Ka is compact and semi-Dedekindian (*351'11). When Kc is a family which is open, connected, and submultipliable, if L e K, and p e NC ind - t'O, we have (AM). Me K,. rep/'M* = L (*3512). Hence if X is any ratio (excluding ooq, now and always henceforth), we have E! X K,'L (*351-21). In order to obtain the same result for K, we have to assume that all powers of members of /c are members of Kc (*35122), but we can obtain the same result for K v Cnv"'K without this assumption (*351'221), because of *331p54, which shows that in any connected family all powers of members of K Cnv"c are members of K u Cnv"K. In virtue of the above propositions, the propositions on products and sums of ratios, which in *350 only stated inclusions, now state identities. Thus if X, Ye C'H', we have (X K,) I (Y r,) = (X x, Y) K, (*351-31), rep, {(X t K4 L) | (Y Kt 'L)j = (X +8 Y) t K,'L (*351-42),

SECTION C] SUBMULTIPLIABLE FAMILIES 419 where L e c,; also rep,'{(X K,'L) (Y e,'L)} = (X -, Y)t,'L (*351-45). The corresponding propositions for ratios confined to K instead of to K, require the additional hypothesis s'Pot"K C K, because this hypothesis is required in *351*22; on the other hand, in the analogue of *351'42 "rep," does not appear, and we have (with the above hypothesis) (X t KR) (Y tc'R) =(X +8 Y) K'R (*351-43), where R eK. For ratios confined to K v Cnv"K instead of to K, the corresponding result can be proved without the hypothesis s'Pot/"K C (*351'431). It will be observed that the hypothesis s'Potc" C K is satisfied if c is a group, though it may also be satisfied when K is not a group. Since a transitive connected family is a group, a transitive connected family always satisfies s'Pot"K C K, as has been proved already (*334'132). *351'01. FM subm = FM n K{R K. vNCind- 'O.)R,V. (S).S K. R= S} Df *3511. F:. ce FMsulbm. =: e FM: R K. v NCind - t'0. R,v. (aS). S K. R= S' [(*351'01)] *351'101. F:! FM subm.. IIfin ax [*351'1. *30116.*300'14] *351'11. F: K e FM sr. Cnv"Ka e comp n semi Ded.. K e FM subm [*337'27] *351'2. F:. c e FM ap subm conx.: eC NC ind - '0. L e K.. (aM). M e K,. rep,'MFL = L Dem. F.*3511. F: Hp.,eNCind-t'O.Q, ReK.L=QI R.. (aS, T). S, T. Q = S. R =. [*332-53] D. (aS, T).S, ' e K. L=rep,'(S I T)': D F. Prop *351'21. F: Hp*351'2. X e CH'.L,.. E! X ic,'L Dem. F. 351'2. 332-61. F: Hp., e NC ind - t'0. X = p/v.. (aM). Me K,. rep,'ML = rep,'L". [*35041-5]. E! X K,L (1) F. *350-34.: Hp.= 0. v e NC ind -'O. X = u/v.. X C. c,'L = I r s'A"c (2) F. (1). (2). * F. Prop 27-2

420 QUANTITY [PART VI *351-22. I-:Hp*35P2.S'Pot"'KCK.XEC'H'.REc.:).E!X~K'R Dem. F-.*301P22.:)F: Hp.u,vENCind.v+0.).-RILEC. [*351-1]).(2[). S EiK. -RA= SP. [*350-4.*331-12].(S).SKSivR(1 F.(1).-*350-521. DI-. Prop *351-221. -: Hp *3-1-2. X eC'H'.X-K vCnv'K. REX.:).E! X ~X'R [Proof as in *351P22, using *331P54] *351-3. I-:Hp*351P2. ~vENCind.v+jO.:). Demn. L {(pIV) ~ Kc1} N..L, N IE K,. repk'LP = repK'Ng. [*351P2] =. (gM). L, M, NE6 K,. L = repK,'M'*. repKIL" = rep,,'Ng. [*333-24] =. (2jM). L, M, NE Kc L = rep,,'MfL. r.epK "M/'x Xrep/NIL.g [*333-44].(RM). L, M, Ne6 K,. L = rep,, Mg' re p, "MP = rcpK"N. [*3.50-41] 2. (NM). L f (,t/1) ~ K~} M. M t(I/V) fi,c NI(1 1- *350-34. D I:. Hp. = 0.): I-.*350-34. *351-21.) I-Hp.p = 0.): F-.(1).(2). (3).:)F. Prop *351-31. 1F: Hp *351V2.X, YE GOH'.). (X Kp)1 (Y 14) = (X X, Y) K, [Proof as in *350-6, using *35313 instead

University of Michigan Historical Math Collection: Principia mathematica, by Alfred North Whitehead ... and Bertrand Russell.

of *35051-3] *351A4. i:KEFMapsubnicoinx./LIV,p,o-eNCind.v=1=.o-+0.LEK,.:). rep,'[(Al) K,'L} {(p/o-) ~ K,'L}] = (ply V p/a-) ~ KLL Dem. F-. *350-41.):Hp. p 0. p # 0. M =(y) K,'L.). repK'MP 1-repKDLg. [*333-44]).rep,,'M"x,<O' rep cILgxcO (1) Similarly F:Hpq.,u 0. p 0. N =(p/a-) 14',L.:). rep,,(NvxcO =rep 'Lv'xcp (2) F-. (1). (2). *333-34. *332-33.) F-: Hp (1). Hp (2).). repc'(MI N)vxc~r = rep,I{L4, xcO) IL("xc)} [*301L23.*333-24]:). {rep,,'(M IN)}"vxr- rrepcL4'Axc~) +c (PxcP) [*306-14.*350,41]:). rep,,'(M IN) = (,ply +8 p/O-) ~ K,'L(3 F.(3). *351P21. *350-34.:) F-. Prop

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SECTION C] SUBMULTIPLIABLE FAMILIES 421 *351-41. F:K EFM fap subm conx. sPot",Ic CK.,at, v,pa"E NCind. v +0. a"+0. RE K -:. J(/P K'R} f(p/G") ~ KR= (IA/V +g p/a") ~ Kc'R Dem. F. *351"21-22.) F. (1). *332-241. *331P24:33.:) F: Hp.). {(~L/v) ~ K'IRJ I{(p/a") ~ K']?} repK'"[{(/./v) ~ ic,'RI I{(p/a) ~ 4,R1] [*351-4. (1)] = (/V +, p/a) ~ K'R:) F. Prop *351-411. F:Hp*3a51-4.=KV CnV"K.SEX.:). L) {(Ft/v) VRX'SI (p/a") ~ X'S} (O4V +.q a/c") ~ X'S [Proof as in *351-41, using *331P54] *351-42. F: KeFMapsutbm conx.X, YEC'H'.LE6K,.). rep8'{j(X ~ L'fL) (Y ~K1'L)} (X ~., Y) ~ KL [*351-4] *351-43. F KceFMap sutbm coux. SPOt"K C/K. XI"EC'H'.REK.:). *351'431. F:Hp*,351-42.X=KvCnv"K.SeX.:). *351-44. F K cFM ap sibmconx. LIYa" e NC ind. v tO0. a #O. (p/o-)H'(At/v). LE6K,.. rep,,[f(/I/v) r K'LJ I {(p/O") r K,'JL} I (F/v -s p/a") r KLL Dem. As in *351-4, F:Hp.- M = (Ft/v) r K,'L. N = (p/a") r Kc,'LJ frep8'(MIN)},,xe, - rep,1LIS*xCT LYXCPI (1) F.*301'23. *308&13.:) F: Hp. 7-= (At x(, a") -(v x0, p).) ropK gIxca LPXcPJ - 'ep[LT ILvXcp IIXcP} [*72-59.*332-25] = repK/L (2) F. (1). (2). *350"41.) F: Hp (1). Hp (2). D. rep8,'(M IN) = IT/(v X,, a")} ~ K,'L F. (3). *308'24.) F. Prop (3)

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422 QUANTITY [PART VI *351-441. F KcE FM ap subm conx. ""VI PI pa' E NC nd. v+0 - u-+ 0 (tzv) H' (p/u). LfKE rep'[Iu/) K4'L} t(p/Gu) ~ Kc,'L}] = (O./v -sp/u) K~t Dem. F. *332-15. *303-19. F: Hp's). rep5'l[t(,L/v) ~ Kc4'L} t(p/u) ~ KjtL}] Cniv'rep5'[{(p/au) K 14L} I (,u/V) ~ K~'LJ] [*351P44] = Cnv'(p/o- -,,UItV) K,14L [*303-19] = (p/u8 F'/V) K," [*308-21] = (PjV sp/u) K,,"L: F. Prop *351-45. F: KceFMapsubm conx.X, YeC'H'.LeK,.:). rep5 '{(X ~ KC,'L) I(Y~ ~ K~L)} (X -sY) I K4'L Dem. F. *3.5121. *350-34.*30812.:)F:Hp.X= Y.) rep5'{,(X K i4L)I (Y ~ K,'L)} = I [r s'W'K = (X -s IF) ~ K,4'L (1) F. (1). *351V44441.) F. Prop *351-46. F: KE FM apsubm con x.SPot "'KCCC.X, Y cGOH'.-Rex.)K Dem. F. *351-22.:) F: Hp.). X~ FC'REIC. Y~,c'1RE K. [*37-62] D. X K" ICBI K. Cnv'Y~ i Rc'RCnv",ic: D F. Prop *351-47. F: Hp*351V46.:).(Cn'IFV Y K'R) (X ~K"cR) =(X -s Y) K [*35 1 45A6]

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*352. RATIONAL MULTIPLES OF A GIVEN VECTOR. Summary of *352. By a " rational multiple " of a given vector in a family K we mean, if we are dealing with Kc, any vector in the family which has to the given vector a relation which is a member of C'H', and if we are dealing with K,, we mean any member of c, which has to the given member of K, a relation which is a member of C'Hg. We will call the former "rational ic-multiples" and the latter "generalized rational multiples." It will be observed that if K contains pairs of members which are each other's converses, only one member of such a pair can be contained among the rational K-multiples of a given member of K, provided Kc is an open family. Hence the rational c-multiples of a given vector all have one "sense," even if this was not the case with the original family. Rational multiples of a given vector 1' can be arranged in a series by correlation with their measures with T as unit. These measures are ordered, in the case of rational c-multiples, by the relation H', and in the case of generalized rational multiples, by the relation Hg. Moreover if X is the measure of a given member of Kc with T as unit, the given member of K is K1 AT'X; while if X is the measure of a given member of K,, the given member of K, is, 1 AT'X. Hence the rational K-multiples of T are ordered by the relation K AT;H', and the generalized rational multiples are ordered by the relation K,1 AT;Hg. These two relations, therefore, are the relations we shall consider in this number. We put *352 01. T, = c 1 AT;H' Df *352 02. T,, =, 1 AT;Hg Df We assume throughout this number that K is open and connected. In dealing with T,, we assume TeicK, and in dealing with T,, we assume Tec,a. We then prove the following propositions among others: 1 AT r C'H' e 1- (*352-12), K, 1 AT C'Hg e 1 - 1 (*352-15), i.e. the relation of a rational multiple of T to its measure is one-one.

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 T,, T,, e Ser (*352'16-17). Observe that this requires only that K should be open and connected. The serial property results from the correlation with H' or Hg. C'T, = K n AT"C'H'. C'T,, = K, n ATCW'Hg (*352-3g31). If Sis any non-zero member of C'T,, C'S, = C'T, (*352-41), i.e. the rational K-multiples of T are the same as those of any rational K-multiple of T; with a similar proposition for C'TK, (*352'42). RT,S.: R,Se K n ATC'H': (Ef[,v).q/,V e NC ind.L < v.R- = S' (*352-43). This is a convenient formula for T, and leads immediately to T, = {'H"(1/1)} t (K n AT"C'H') (*352-44). Observe that H"(/1) is the class of rational proper fractions, including Oq. By *352-44 and *352'41-3, we see that, if S+ I r s'(I"K, S C'T,.). S,= T, (*352-45), i.e. the order of magnitude of a set of vectors which are rational K-multiples of a given unit is independent of the choice of the unit. In order to establish the analogous property for T,, we first prove a formula analogous to *352'44, namely T = Cnv;{1'H'(1/1)} t (, nA TA"C(H) {'H"(1/1)} t (c, n AT"C'H') (,352-54). Here the first term gives the series of negative multiples of T, while the second gives the series of positive multiples of T (including I r s'(I"c). From the above formula it follows, as in the case of TK, that if S is a positive multiple of T (not including I s'(I"cc), S,= T,,, while if S is a negative multiple of T, S.,= T,, (*3525657). Finally we deal with the relation of U, to T,. Here we have to assume that K is a serial family. We then find that U, with its field confined to rational K-multiples of T is the converse of T,, i.e. we have *352-72. F: K eFMsr. T e ca.. U cC'T, = K 1 AT;H'= T *352-01. T=K1 AT;H' Df *352-02. T, = K, 1 A;Hg Df *352-1.:. RTS. -: R,S e: (IX, Y). XH'Y. RXT. SYT [(*352-01)] *352-11.:. RT,,S. -: R, Se K,: (gX, Y). XH Y. RXT.SYT [(*352-02)]

SECTION C] RATIONAL MULTIPLES OF A GIVEN VECTOR42 425 *352-12. F-: KeFMapconx.T TKa.). K1ATr C'H',eI-+ I Dem. F. *336,1. F:-:R (iclA~rrCH') X.E =. RCEK.XEOH'. RXT (1) F. *350-1521.)F: Hp. R,f~EK. X eO'H'. RXT.-SXT.D. R =S (2) F.*350-44.)F:Hp.Re/ca.X,YEC'H'.RXT.RITT.).X= V' (3) F. *35"0-34 4.:) F: Hp.R=~Ir S 6'GK.X,YeCOH'.1?XT. SIT.)D.-X =0Oq Y=0Oq (4) F. (3). (4).)D-: Hp. REK. X, YeCOH'.RXT. SYT.D. X =YI (5) F-. (1. (2). (5).) F-. Prop *352-13. F: K eFM apconx. TeK,a.:). K~ ATA6P'f C 1cia Dern. F. *350-4.) F: Hp. R E K, A AT"C'H.)D. (gz i). p, v c NC indi - t'0. ft! RI' Ai P. [*333-101] -R. 1?e,a::) F. Prop *352-131. F:Hp*352*13.).K ric ATA 10Hn, = CIIV"(K, A AT"C'H) [*307-1] *352-132. F: Hp *352-13.).K,,n A 6"CG'Hl C K, [*352-13-131] *352-14. F:1KceFMap conx. T6K.).K rc, A A"CH AT"C'H? = A Dern. F.*.307-1.*350-4.*352-132.:)F:Hp.R, SEK,.REAT" 6C'Hf,. SEcA T"G'CH'.) v~, V)P, O-). /a, v, p, a- e NC ind. v # 0. p # 0. a- 4 0. RC K a rep, IRv - repK/TIS. repK/S'T repK'TP. rep,,IR~xcP =rep,'T,,tx(P rep, SxIL. [*333-101.Transp] D. R + 8:)D F. Prop *352-15. F: K EFM apcoInx. TE c 1l K, '1 AT rC!'Hql 41I-+ Dem. F.*336-1 F): Hp. R (K, 1 A T U'flg) X. R (K, 1 A T rG'Hq) Y.)D RE K,. XYeGC'l~q. RXT. R YT(1 F.(1). *352-14.) F:. Hp (1).)D: RE6K,. X) YE C'H'..XT. RYT. v RE6K,.-X, YeCOH,. RXT. RYT: [*307-1.*350-44. *352-13'132]): X = Y (2) F.*336-1.) F: Hp. R (K, 1 AT r C'Hg) X. S (K, 1 AT r C'q) X.) R, SE6K,.X EG'Hg. RXT. SXT. [*350-521.*307'1]:).1 =8S (3) F.(2). (3). DF. Prop

426 QUANTITY [PART VI *352-16. I-: K eFMap conx.TeKa.:). T, eSer [*.352-12. *304-48] *352-17. F: K c FM ap conx. e TE K). T,,, e Ser [*352-15.*307-45.*304-2.3] *352-18. F: KeFMapeonx.S'POt'%cK C~ca.K Cnv"6Ka=A. Telca.:) Kmf AT6'CH,, = A Dem. F. *350'43.. F:. Hp. p, ve NC ifld - t'O. X = (p~/v) ICnv. Se K.:): SXT. =. 5" = Th. [Hp] D. S"e6Ka ACnv"Ka(1 F. (1). Transp. D F: Hp.) e(HX, S). Xe6 O'H, SE K. SXT::) F. Prop *352-181. F: KcFMimit.TEKa.:).~n AT"C'Hfl,=A [*352-18.*335-21] *352-2. F: KeFMapconx. TeKa.).(ItS'Ul"K)TKT Dem. F. *350-34. *331,22.)3 F: Hp.). (I r' S'U"I~K) 0q T F. *304-45-48. DF: Hp.).-0, OH' (i/l) (1) (2) (3) *352-21. F: KeCFMap conx.The,a.:). (I r S'G"K) T,,, T [Proof a *352-22. F:Kc FM apconx.TE Ka.~T, [*352'2 -*352-23. F: Ke FM apconx. TE K~.)a!Ta [*35D2-2 -*352-3. F: KeFMapconx. TE Ka.. C'T,= KA% AT"'C'GH' Dem. F. *350-31. *304'48.) F:Hp.XeC'H'.X+1/1.).X(H'wH')(1/1).T(I/1)T. [*306-1]).X e (H'w Hw "lc F. *350-34. *331P22. *304-45-48.) F: Hp. X = 1/1.).XH' Oq. (I r S'G"K) 0q T. 1Ir S'(J"K C K. [*306'1]). X H"'A(tic F.(1).(2).)F:Hp.).CH' C(H' " I')"A ccK F. *1 50-201.) F: Hp.:). C'TK = KJ1ATC"(IIf'V'H')"AT'K. [(3)]:). Kj1 AT"'P'H'C C ~'Tf F.

(4). *150'202.) F.Prop Ls 1 ~1] in *352-2] (1) (2) (3) (4)

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SECTION C] RATIONAL MULTIPLES OF A GIVEN VECTOR 427 *352-31. F K E FM ap conx. Te IctKa). CT.4 = K~ n AT'P('fHg Dem. As in *352-3, F-: Hp.)D. G'H' C (Hg, W Hg)" AT"cK (1) [*336-1] D. XeHg"ATccK (2) F(1).(2). D F: Hp.:). C'H C (Hg wJH)"A t"K (3) F. (3). *150-201-202.:) F-. Prop *352-32. F:. Hp*352-3.X, Ye6CH'.R=X~ K'T.S= Y~ic'T.): RT~S. =. XH'Y [*352-1. *350-521] *352-33. F:. Hp*3.5231.X, YE CHg. R=X~K".=~,T RTKLS -. X4,q Y [*352-11-15] *352-34. I-:. Hp *352-3.D: RT T..(X). XH'(1/I). R X K'T [*352-1. *350-521-31] *352-341. F:. Hp *352-3.D: TTKR. ~(gX). (1/I) H'X. R X KcT *352-35. F:. Hp *352-31. D RTIcT. (HX). XHg (I /I). R X K,'T [*352-11-15] *352-351. F:. Hp *352-31. D: TT,,R. (jX). (1/1) HgX. 1? X K,c,'T *352-36. F: Hp *352 3. s'Pot"Kc C K.) Pot"'T - t' T C TK' T Dem.. F. *3 5043.:) F: Hp.v eNC i nd - t fO- t'I) I T"(v/ 1) T. [*304-4.*352-341]). TT, T"::) F.Prop *352-37. F:Hp *35-2-31. TeK v Cnvic'.).Pot'T - 'T C TK'T Dem. F. *331'24'54.) F:Hp.). Pot'T C K Hence as in *352-36. *352-38. F: Hp *352-31.) rep,,"(Pot'T - t'T) C Tc,'T Dern. F. *332611. F: Hp. rep,,"(Pot'T - t' T) C K, Hence as in *352-36. *352-41. F:KeFMapcoDx.S,TfKa. SeC'Tc.:). C'S~c = C'T,, = K ri AT'P'H' - Kc n As"G'H Dem. F.*352-3.*350-43. 2 F: Hp.).(gp~tv). p,v~eNCind-t'O S*T" (1) [*352-3]:). T eC'SK (2)

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428 QUANTITY [PART VI F.(1).*352-3.*350-43. DIF:Hp. Re C'IS,. (,v0p) j.P,V,o e cNC ind - "O. p e NC ind. S9 = T" I - P [*301)504]). v pa-.1, v, a- e NC, i nd - tf0. p e NC mdR~xc-A- Txp [*332-3.*350-43]) R B CT (3) T S I -.(3).(4).*315 2 - 3.)I. Prop *352-42. F: KCFMapconx.S,T TK~a.Se C'TKL.)*SKL C= K Dem. I-. *352-3. *350-4. *307-1.):Hp.) v:,a, vC6NC ind - t:O SvAt~TfA.vl! ATs (1 [*352-3 1] ITe (J'& (2) F. (1). *352-3. *350-4. *3071. D F:. Hp. 1?e CC'SK) (a[~I VI PI a-): Fv, o- c NC ind - tf0. p C NC nd: Aj! Sv A% Th. v. A! S", A TfL: Ra AN SP. v. f! R" A: SP: [*333-48])(a/, V,v,p or):,uv,o-e NC ind — ft0. p cNC ind: i R~xc/LA %T,,x! P. v. Aj! Raxl A %TvxcP: [*3152-31]): R I? T (3) F(2). (344.)S3F::Hp. R cCOT,.). R cC'S,,, (4) F. (3). (4).:) F. Prop *352-43. F::KeFMapconx.TeKa.):. RTKS.=: R,Se K^nAT"'H': (,v).,Ft, Pe NC ind. /' < v. R'- S Dent. F. *33-17.F: RTK~S.2R, S eC'TK. RTS (1) F. (1).*352'3-1. *350-43.:)F:: Hp.)D:. RTKCS.:R, SC rn AT"C'H': (e a,).- 4qcNCind -t~O.pc~NC ind. p x0,, < a xC4:. Ra"-TP. S-q:T V [*333-5] R, S CK r% ATCH: (1p, a, 4:,).o4qc N C id - t'O. pe NC i d.p x, i < a xc 4~* Rx = TPxc~t = SPxc-: F. *350-43. *304-4.) F:. B, SCK nAT"IC'H': (gtpv).FtpCvNCindi. Ft< v. B"=S:: R, S C K rn AT"f'~'H': (gX). XH' (1/1). RXS: [*336-1]:):R, SCex:dX, Y,Z). XH'(1!1). Y, ZeCH'.RXS. RYT. SZT: [*350-6.*.305-7151]:): R, SC Kc: (a[X, Z). (X x8, Z) JI'Z. R (X x, Z) T. SZT: F. (2). (3).:) F. Prop

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SECTION C] SECTIN c] RATIONAL MULTIPLES OF A GIVEN VECTOR42 429 *352-44. F:KEFM ap conx. Teva.).T f='II"(1/1)} ~(KcnAT""' Dem. F. *352-43. *304-4.:) F:: Hp.):..RTKcS.: R,SEK cA A"C'H': (gX). XI'(1/1). RXS::) F. Prop *352-45. F:K EFMapconx.S,T1~ c.Se 'T,.:). S,= T, [*.352-44-41] *352-5. F: KeFMapCoHx. [FE ~a.). C'<~ AT;H'=KLAAT"G'"H' [Proof as in *:352-3] *352-51. F: ic FM apconx. TEKd Ict C G',, 1 AT;iII17t= K,c eAT"G'II?L Dern. F. *150-202.) F: Hp.:). GC'K,1AT;H,,C K, AT'P'CHI (1) F. *352-131.)D F: lip..R EK1c AT"CG'H,,.).(aX). XEC'H.-REKL.RXT (2) F.*304-23.)F:Hp.XeC'H-t'(I/1).REK,.RXT.:). [*307-1.*336-1] DN.R6C GKJ1AT;H~ (3) F.*352-38.:)F:Hp.X=1/1.RE K,.RXTI.)R (clAT;H) (rep,, IT2). F.(3).(4). DF:Hp.XEG'lH.RC K, RX T..R eGOK 1 A TUJ (5) F.(2).(5).:)F:Hp.:). K1rI~AT'lC'fl,,1C0'11, A T;Hn (6) F.(I). (6). DF. Prop *352-52. F:/KEFAlapcotix.Tc/ a.:).T, ~K= KAT;Hn t-K~I1AT;H Dewi. = Cnv;K, lAT;Ht4~K,1 AT;H' F. *160-43. (*307-05).)D F. T,=I '41 A~H,,K,J 1A;H'vi~(KJA ArCCCH,) t (KJA~"" 1 F. (1). *352-551. *307-1 D) F. Prop *352-53. F:Ke cFM apconx.TeK1o.). K~ ATW = {kl'H"(1/1)} (K,nA AT"C'H') [Proof as in *352-44] *352-531. F: Hp*352-53.D). KJAT;H- {h'rHf(l/1)} ~ (K,nAT'CH [Proof as in *352-44] *352-54. F: Hp *352-53.).T,= Cnv;{h'H'(1/1)} ~(~, At-AT"C'H) tlh'H"'(1/1)} ~ (K,nA AT"C'H') [*352-52-53531] *352-55. F:KEFMapConx.S,TEK~a.SEKriAAT"C'H.) K1 r A sCH= K,^r AT"G'CH'. K1n Asl C KH = c, AT"GC'H [Proof as in *352-41]

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430 QUANTITY [PART VI *352-56. F:KEFMapconx.S,TEC Ka. SE K, nA "W'H. -) -. =T [*352-54-55] *352-57. F:iceFMapCOnx. S, TcK~.a -SeK,4r ATW'IHfl.:). S, = T., [*352054-55. *307-1] *352-7. F:. EFM sr. X.YeCH'H. TIEKa. I',Qe K. PXT. QYT.: P-U,, Q. -. X H'Y Demn. F. *352-18. F: Hlp. Q!PE D.)Q P,,e AT"'Hfln [*350i65] D. X -,Ye C'H' F. *350-52. DF: Hp(1).)X=[Y [*308K12-19.Transp] D. XH'Y F.*336-64. D F:. Hp. r (PUKQ).): P =Q. v. Q U.P: [*350-44.(3)]): X = Y. V. YH'X: [*304-48] D:r(XH' Y) F. (3). (4).)D F. Prop (1) (2) (3) (4) *352-71. F:. KCeFMsr.TIE/a.P,QeC'TIC.):PUKQ. =. P(ATf;H') Q [*352-73] *352-72. F:KEFMsr. TIE~c.:).U4,C'T-,KJAT;H'=TK [*352-71] *352-73. F:.Kie FMsrsubm.XYeCI'H'. TEC~a.): (X~,c'T) UK(Y~ ~ KT).. XII'Y [*30527. *351-22]

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*353. RATIONAL FAMILIES. Summary of *353. A "rational family" is one which consists entirely of positive rational multiples of one of its members. We denote rational families by "FM rt"; the definition is *353-01. FMrt=FM m {([(T). Te ca. K C AT"C'H'} Df It is obvious that, if K is any family, K ^ AT"GC'H', which we considered in the last number, is a rational family. If K is a connected family, it does not follow that cK AT"C'H' is a connected family, but the proofs of its properties, as we saw in *352, make use of the fact that it is contained in a connected family. Many of the most important properties of connected families hold equally of sub-classes of connected families, notably the property that two members of Kc or K, whose logical product exists are identical (*331'42'24). In dealing with rational families, a good many propositions can be proved by merely assuming that they are contained in connected families. We put *353-02. FM cx = FM n X {(2tc). K e FM conx. X C K} Df *353-03. FM rt cx = FM rt n FM cx Df We will call a family "sub-connected " when it is contained in a connected family. When a family K is open, rational, and sub-connected, any member of Ka may be taken as the T of the definition *353'01 (this is proved in *353*13); and if S, T are any two members of Ka, some power of S will be identical with some power of T (*353*12). An open rational sub-connected family is asymmetrical (*353'2); no power of a member, and no product of two members, is the converse of a non-zero member (*353'22'23). Hence by *331P54'33, if the family is connected, and not merely sub-connected, it is a group and transitive (*353'25'27). If X is a family which, besides being open and rational, has connexity, then if a is a member of the field and T e Ka we shall have s'xa = Aa;X 1 A;H'. UA = x 1 A; H' (*353-32-33). That is, the series of points in the field and the series of vectors are both

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 ordinally similar to part or the whole of the series of ratios; they will be similar to the whole if X is submultipliable (*353-44). But when X is submultipliable, a smaller hypothesis suffices, for in that ease we can prove that if X is connected, then Xt X v Cnv"X (*353-41), so that X has connexity, and is serial (*353-42). Thus we have *353-44. F:Xe FMapeconx rt submi.).~'Xa sinor H' *353-45. I. FM ap conx rtsubm CFM sr *353-01. *353-02. *353-03. *353-1. *353-12. FMrt=FMtK'[(HT) -TEKa. KCAT"C'H'} D f FMcex=FMnX{(aK).Ke FM conx XC K} Df FMrtqx= FM rtriFMcx Df I.:,c FM rt. =:KIe FM: (aT). Te Ka. KC CAT'"C ' [(*353-01)] F:XceFMap rtex. S, Te X. XC AT"GH'.3. (gjt, v).,. v cNC ind -v +.Sv, =Tf [*350-4:3] *353-13. 1-: X e FMf ap rt cx. T cX-.) X CA '"G'H Dent. F - *353-12.:) I-:Hp A SEX X C As"C'H' Be (3/L, v,p, aT) -p~,v,p, a-cE NC ind p +0. v +z0. a-+ O. =St h [*333-5] - (aji'1 v,p, a-).,t, v,p, a-c NC mnd. p+ 0. v zzO.o- +0. BvxcP= - ixcP=- Thxcc[*3.50-43] D. I? AT"G'fl'H:) F. Prop *353-14. 1-: Hp*353-13.). X4CA A'"GH Dem. F.*353-13.)F: Hp.B,SeX)D.(a1X, Y).X, YeC'H'.RXT.SYT. [*3.50-65] D. (R IS)(Y -8X) T. [*308-2] D. RIjSeA T"G'Hg:) DF Prop *353-15 F: KcFffconx. TEKa.:) KcnAT"C'H'cFMrtcx [*353-1. (*353-02)] *353-2. F: XecFM ap rtcx. D. X n Cnv"'Xa= A. XecFMasym Dem. F.- *35:3-12-13.-) F.*333-101.:) F: Hp (1)) Pot,'R C RJ (3) F. (3). (2). Transp.- (*334-05).)F. Prop

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 *3O53-22. I:Hp *353-2.).s' Pot"X a Cnv""Xa=A Dem. F. *353,1213.*30P5.) F: Hp.o-ENCind-t'0.R, RoEXa.). [*301P23]:). (Hp v). p, v E NC ind - t'0. RIL+c(Oxcv) C I (1) F.*333-101.*330-23. DF:Hp. D.Pot'R CJ. A e EPotR (2) F.(2).(1).Transp.)F:Hp.REXa.).cg~! Pot'RnCniv""X::).Prop *353-23. F:Hp *3532.).(s'X II. "X) CnvlXa = A [Proof asin

*353-22] *353a24. F:Hp *353-2. XE FM conx.). s'Pot"X CX [*353-22. *331P54] *353-25. F: Hp *353124.:). s'\ " 6X C x [*353-23. *331-33] *353-26. F: Hp *353-24. D. s'Xa "6Xa C Xa Dem. F. *353-12-13.:) F: Hp. R, SE k* X. v). /.t, v e NC ind - t'0. =I [*330507]:). (vet, v). p, v E NC ind - t'10. (RB S)v = S&+-v. [*333-101] D. al! Pot'f(R IS)v rRLJ. [*301P3.Transp.*331P23]) I S ERV"J (1) F. (1).*3053-25.) F.Prop *353-27. F:Hp *3-5324.). X EFM trs asym [*353-26'2.*334-13] *353-3. F:. Hp*353-2.vENCin1 - t'O.s'Pot"X,CX.):RUAS.).R kS Dem. F. *336-4l.:)F: Hp.:). (T). TEXk.R=TIS. [*330-57])(aT).TE xa. RV = TPSP [*336-41.Hp] RI' UkSP::) F.Prop *353-31. F:.XEFMaprtconnex.R,SEX.vENCind-t'0.): R(AxS. R.US Dern. F.- *336-62.) F: lip. R#. (RUkS) StkR. [*353'3-24] SP UkR,. [*336-661. *353-27] D (RV UkP~) (1) F.*336-6.:)F:Hp.R=S.).e-..(RUAS,,) (2) F. (3). *353% 3. D F. Prop R.& W. Ill. 28

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434 QUANTITY [PART VI *353-32. I-:XVeFM ap rt connex. TE Xa.). (= X1A;H' Dem. K*353-1213.-*350-5)I: Hp. R, SeX. Rt+S.) F.*301-5.)D F:Hp (1).~, v, p,ae NC ind. v +O.oa- O. RP= T9 Sa= TP.I x(a- < vx,,p. RvxcTr svXc~ = T(vxco) -c (Izxc'Ti (3) I-. *334-21.) I- Hp (3).:).R ISE X vCnvl"X. r*331P54.*332-241]).(R IS) VXC47 rep, '(1 S)Vxco[*332-53.(3)] =T(Pxcp) c4fLxca) (4) F -.(4). *353-24-2.:) F: Hp (3).. I S ~X [*336'41]).SUCR (5) F.(I).-(5). *304-4.) F:Hp. R(X 1AT;H') S.:).SUKR (6) [(6)])RUKS. [*336&661.*3,53-27]). (S U"11) (7) F. *336-6.)D F: Hp. 1= 8.). (S UKR) (8) F.(6).-(7).(8).) F.Prop *353-33. F:Hp *353-32. a 6s'U"X.). ~4'=Aa;X1AT;H' Derm. F. *336-43.)F:Hp.) U, =X 1 Aa ~'Xa (1) F.(l). *336-2.:)F:Hp.:).'Xa=Aa;Ux (2) F. (2).-*35:3'32.) F. Piop *353-34. F.FM ap rt onnex CFM sr [*353-27] *353-4. F:XEFMaprtcx.,s'Pot"~,'CX'.LeX~a.). (Hjo-). a- 6 NC hid - tf0. repA'L1T 6 X Cnv"X Dem. F. *353-12,1.3.) F *301-23) [*332-53]). rep,,"(R fS)"E e (2) Similarly F. Hp (2)'): > v.).rep,,'(R IS)ILE Cnv"ic (3) F. (1). (2). (3).:) F. Prop

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SECTION C] RATIONAL FAMILIES 435 *353-41. I-:X6FM apeonx rtsubm.).X, =XvCnv"'X Dem. F. *353-4. I-: Hp. L e X~.).(HR,oa-).R E X uCnv"X.oa-,NC ind-ti/ O rep,,'I~ r=R' [*333-41]:). L X v Cniv"X,:) F. Prop *353-42. F: Hp*353-41.).X FMs,-r [*353-41.*334-26.*353-27] *353-43. F: X EFiap cx rtsubm. Te Xa.Potid'TC X.).G'H' C AT6"X Dem. F.-*3511I.) F:Hp.,vENCd.v 0.).().S\.5 -T [*350-43]1) ([S). Se X.- S(I/v), T (1) F. (1). *336-1.)F:Hp. XEC'H'.). (2S)-SEsX. SATX:)FI. Prop *353-44. F: XEFM apeonx rt subm.).- 'XasmorH' Dem. F. *353-42-33.: F: Hp. aes"(,'WX.).g"'X=AaX1JAT;H' (1) F. *353-43.:)F:Hp (1).-).G'H' C J"(A,,,IX1 A) (2) F.*336-2*352415.)F: Hp(1).D.AaIX1A~rC'H'e1 ---1 (3) F.(1).(2).-(3).) F. Prop *353-45. F.FM apconx rt subin C FJsr [*353-42] 28-2 9

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*354. RATIONAL NETS. Summary of *354. The subject of " rational nets," which is to be considered in this number, is of importance for the introduction of coordinates in geometry. We have three stages in the construction of a rational net. First, taking any vector T in a family K, we construct C'TK, i.e. the positive rational multiples of T, as in *352. The result is, as a rule, a family which is not connected, even when the family K is connected. For if there are in K any vectors other than C'T,, any point of the field which is reached from a given point a by one of these " irrational" vectors cannot be reached from a by a member of C'T, though it will be in the field of C'T,. Thus in order to obtain from C'T, a connected family, we shall have to limit the fields of its members to the points which can be reached from a given point a by one or more rational steps backwards or forwards, i.e. to the points A,"(CG'T,),. It will be observed that whereas, in the construction of C'T,, only positive vectors are used, negative vectors, i.e. the converses of positive vectors, are also admitted in constructing what we may call the "rational points" with respect to a and T. Having constructed these points, i.e. the class Aa"(C'T,),, we then proceed to the third and last stage in constructing a rational net, by limiting the field of every member of C'T, to A,,(C'T,,. Many of the propositions concerning rational nets require the hypothesis that the family concerned is a group. If this is not the case with the family K from which we start, we replace K by Kg, where Kg is formed by adding to K the converses of those members of K (if any) whose domains are identical with the common converse domain of members of K. The definition is *354-01. Kg = K V Cnv"(K n D's'("K) Df We put also *354'03. FM grp = FMn^ /(s'K J"I C C) Df We then easily prove that if K is connected, Kg is a group (*354'14), and if K is open and connected, Kq is open and connected and a group (*354'17). If K is connected, (Kg), = K, (*354'15), so that properties only dependent on K,, like that of openness, always hold for Kg when they hold for K.

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SECTION C] RATIONAL NETS 437 Next, we prove that if Kc is open, connected, and a group, C'T, is open, rational, sub-connected and a group (*354'22). Hence if K is open and connected, and = Kg, C'T, is open, rational, sub-connected and a group (*354-24). The "rational points" with respect to a and T are A"(C'T,,),. In order to study them, we consider Aa"X,, where x is a family concerning which we make hypotheses which will be fulfilled in the case of C'T,. We prove that if X is a family which is a group, and S e X. a e s'(I"X, then Aa"X, C S"Aa"X, (*354-31), whence S (A X,)= (Aa"X,) 1 S= S (Aa"X,) (*354-312). Next we prove that, with the same hypothesis, if b is any other member of Aa"'X,, then A "X,= Ab"X, (*354-33). Thus the rational points with respect to a and T are the same as the rational points with respect to b and T, if b is one of these rational points. The "rational net" is the family t {Aa"(C'T,),}"C'T,. Writing X for C'T,, this becomes: (Aa"X,)"X. In order to obtain the properties of the rational net, we therefore continue to consider a family X, concerning which we make hypotheses which are verified in the case of C'T,, and we put *354-02. cxaX = (Aa"X,)"X Df Thus cxa'C'T, is the rational net defined by K, T, and a. We prove (*354'4) that if X is a group, cxa,' is a family whose field is Aa"X,. We prove that if X is a family, and a a member of its field such that any member L of X, for which La exists is a member of X u Cnv"X, then a is a connected point of cxa'X, i.e. *354-32. F: X e FM. a e s'(". X, n a'A C X u Cnv"X.). a e conx'cxa'X The hypothesis X,n (AaCX vCnv"X would be verified if X were a connected family and a were a connected point of X. But we want to be able to replace X by C'T,, which is in general not connected. The above hypothesis, unlike X eFMconx, is satisfied by C'T,, provided c is open and a group and a is a connected point of K (*354'34). Hence it follows that if K is a family which is open, connected, and a group, and a is a connected point of K, CXa'C'TK is open and connected, and a is a connected point of cxa'C"T, (*354'401). Again, in virtue of *354'312, if X is a family which is a group, and a is any member of its field, cxa'X is a group (*354*313); hence when K is a family which is open, connected, and a group, cx,'C'T, is a group (*354'402); and it is easy to prove that it is also a rational family (*354'403). Hence, by *353'27, cxa'C'T, is a family which is open, connected, rational, a group, transitive, and asymmetrical (*354'404). If our original family is open and connected but not a group, we only have to S

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438 QUANTITY [PART VI substitute K~g for K, i.e. putting X =Kg, we only have to take Cxa'G'TA, in order to obtain a rational net with all the above properties. This is stated in the proposition *354-41. F: KeFMapconx.TcKa.aeconX'K.X Kg.). CXa C' TA e FM ap con x rt trs asyrni *354-01. Kg =K v Crnv"Qc(A D'sWI"'K) Df *354'02. CXaL'X = (Aa"Xt,)"X J)f *354-03. FM grp = FM n K' (S'K I ' C ic) Df *354-1. F:.Reicg.=~:ReKc.V.ReCK.W[R=s'GT"K [(*354-01)] *354-11. F:KeF~tconx. RSE/.:).RjSeKg [*331-33.*354-1] *354-12. F: Hp *354l11. D'I? 8U"K. R IS=5 R. RI A EKt, Dern. F.*330-52.) F:Hp. ae conx'K.).E! R'S'a.P(1(R I)-=s'(I"Ki. [*331-11-42] R I1 SE6 K v Cnv"11K. t(I'(R I5) =8'tI"K. [*354-1.*3303561])R SE6Kg. S IR =RIfS: D F.Prop *354413. F: Hp*,35411I. D'R=D'S=s'G""K.).R SEsKg Dem. F. *31P33. FHp..1? SE"6K WCnV "K(1 F.*37 323.)DF Hp.)D. G(R IS) = S'P'K (2) F. (1). (2). *354-1.) F. Prop *354-14. F K e FM conx s) 8K~q. 66Kg C Kq([*354-11-1213-1] *354415. F: K E FM conx (Kg.), = K Dem. F.*354-1.)F:Hp.R, Se Kg.) R,SE6K.V. R,SEK.V.RSEK.V.ISEK.P[R=I'CS=S'P'cK (1) F. *330-4.:)F:Hp.R,SCK.:). RjSEK, (2) F.*331-33-24.)F:.Hp:RSEK.v.R,SEK:).RISEK, (3) F. *354-12.)F:Hp.1?, SEK.A IR=(I'S=s'P'iCK.)R SEK, (4) F.(1).(2). (3). (4).)FProp *354-16. F: KeFMconx.)./Kg eFM cofx j*354-112] *354417. F: KEFMapconx.). KgeFMapconxgrp [*354-16-1514. *333-101]

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SECTION C] RATIONAL - NETS 439 *354-18. F:.KEFMorp'.-:KcFM.:R,SEK.:)R,S. RlSEKc [(*355403)] *354-19. I K E FM grp.:). s'PotIlc C K [*35418. Induct] *354-2. -: K FAIap conx. TEcKa) C'T,,e Fifap itex [*353-15. *352-3] *354-22. F: K6FM ap coixgrp. TE Ka. C OTKEFM apAtcx grp Dern. F.-*350-62. *354-18.) F: H~p.

R,8, T EKX, YEC'H'. RXT. SFIIT (R S) (X +s Y) T.RJISEcK [*306-67.*3.52-3].R Se C'TK 1.(l). *352-3. DF:Hp. R, Se'TK. D. R ISEC'TK (2) F(2). *354*2. DF. Prop *354-23. F:KeFMit COnx.TEKa.). 'TK= K [*153-1:3.*52-3] *354-24. F:K e FM ap coiUx. TE K-. XKc.:). 'TA E FM a,,p rt, Cx gup. [*.354-22-17] *354-31. F: X EFM grp. a es'C"X4>. SEX.). Aa"4X, C S"IAa""Xi Dern. F.*3361l.)F:. Hp.): xeAa"X,.). (HP, Q).P, Q EK*X = P' Q'a. [*330.56]). (HP, Q). P, Q E K. S'X = P'S'Q'a. [*3-5 418]).(qP, R). P, RCE K. _~ - P'IR". [*336-1]).S'.x E A a" XL. [*37-106]). X S"IA a"Xc:) F. Prop *354-311. F: Hp *35'4-31.).S"Aa"X, CAa"XL [*3.5431] *354-312. F: Hp *35 431.. S ~ (A a"X,)= (A a"Xt) JS = r (A a"XL) [*354-31-311] *354-313. F:XEE M grp.aes'(.I"X. = cxa'"X.).s' "At C,a Dern. F.-*354-312. D F: Hp. 1?, SEX.:). JR (Aa "XX)} 5 (Aa"x,)} =(R 59) ~(Aa"Xt-) (1) F. (1).*354-18.) F:Hp RSE.) { ~ A5"x~} S~(A~")} E cxa'X:) F. Prop *354-32. F: X EFM. a Es"XA. X, ri41'A, C Xv Cnv"X.).a(E COIIX'CXa'X Dern~t. [Hp])(gL) LeX vCnv"'X. x= L'a [*330-43]).(3M).MIECXa'-VCflv"CXa'X x-M'a: [*33111]): a e con X'CXa'X:) F. Prop'

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440 QUANTITY [PART VI *354-33. F X FM grp. a E s'"I" X. b E A,"'X&.) Aax, = Ab"X, Dem. 0 ~~~F. *336-1. F: Hp. c E Ab"X,.). (HjP, Q, I?, S). P, Q, -R, S E K. C =R'S'IP'Q'a. [*330-56]).(HP, Q, R, 5). P, Q, B, SE K *C = R'6P'S'Q'a. [*354-18]. (HM, N). M, N 6-C = M'IN'a. [*336-1].0c EAa6x, (1) Similarly F: Hp. eAa"~11k C). CA b'X, (2) F. (1). (2). D F. Prop *354-34. F:KeFMapconxgi-p.TEKa.X=C'TK. aEconx'c.:). XtAU11aC X v Cnv""X Dem. F. *354-22. DF: Hp.). X cFM apAvcx. [*353-14] D. X, n(K v Cnv"Kc) C X v Cnv"6X (1) F. *331-11I32.:)F:lHp. Lex (fcpAcE.)L 6K VCnV"I P(1) D. L cX v Ctiv""X: D F. Prop *354-35. F:KEFMapcoflx.T EKa. jKgC. XG=CT,. ac EcoiUx'Kc.) A, U'IAa C X v Cnv"X [*354-34-17] *354-4. F: X E FM grp.- a E s'P'IX.) CXa'X E FM. s'Gi"'cxa,'X =Aa,"X, Dem. F.*330-52.:)F: Hp.:).cxa',XCI1-+l(1 F.*354-311. DF:.llp. D:ReXk..'R =Aa6Xi. D'R CU'R (2) F. *354-312. D)F: Hp. R, SE'X. D. {R~(Aa"Xt)U fS~(Aa"'X,)J=(R I S)~(AaXctx) [*330-552] (S R) ~ (Aa"X&) [*354-312] I S~ (A a"XA0)I} {R (A a"'Xt)j' (3) F. (3). *330-5.)D F: Hp.) CXa'X E Abel (4) F. (1).(2). (4) -*330-52. D)F. Prop *354-401. F: KEFMapconxgrp.aCconX'K. TEia.:). CXa'G'T,,cE FM ap coflx. a E coflx'CXa'G'TK, Dem. F *354-422)F:Hp.).cxa'C'T,, cFM (1) F.*354-3432-2.)F: Hp. D. a E conx'cxa'G'Tc (2) F. (1). (2). *333-101.) F. Prop *354-402. F:Hp *354-401.). cxa,, UT, E FMgrp [*354-.313-22A401]

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SECTION a] RATIONAL NETS, 441 *354,403. 1-:Hp *354'401.). X,,C' ePM rt Dem. 0-*5312.*3954-2.) [*354,312.Induct]:). -p v).tv e NC nd. v t 0. {S~ (AafcX,)}v = SP (AaccX) = TI' ~ (Aa""Xi) =f T~ (Aa",X,)}' -L [*a350-43.*30-4.4011:)-(gi v).,u, v e NC nd. p + 0 fS t (A a" X.)j (/4 iiv) fT t (Aa"XA,)J (1) i - (1). *353*1 F.K Prop *354-404. 1-: IEFMapeconxgrp. a econx'c. TEIa.:). CXa'U'T,, e FM ap conx rt grp ti's asym [*354,401'402'403. *35.3-27] *354-41. F-:ceFM ap conx. TeKa. a econx,'K,.X XIg.:). cxct'U'A e FMap conx rt trs asym [*354-17-404]

i~~~~~~ *356. MEASUREMENT BY REAL NUMBERS. Summary of *356. In this number we consider the application of real' numbers to the measurement of vectors in a family. The principle of this application is as follows: If a given set of vectors, all of which are rational multiples of a given vector R, have a limit with respect to U,, and if their measures determine a segment of H, then we take the real number represented by this segment as the measure of the limit of the given set of vectors. For the sake of homogeneity with rational measures, it is well to take our real numbers in the relational form given in *314; i.e. if te C'(, we take s9' as the corresponding real number. With a suitable hypothesis, the result of the above principle for applying real numbers is, where rational multiples of the unit R are concerned, to replace the ratio X by the rational real number s'H'X, as the measure of the vector X Ic'R (cf. *356'63). Then the measure of the limit of a set of rational vectors will be, by our principle, the limit of their measures. Thus our principle is conformable to what is required for an application of real numbers. It should be observed that, if any application of irrationals is to be possible, it is necessary that the vectors of the family concerned should have a serial or quasi-serial order, independently of the order generated by their measures. The order generated, among rational multiples of T, by the ratios which are measures of these multiples, is TK (cf. *352). A vector which is not a member of C'T, cannot be the limit of any set of vectors with respect to TK. But we saw (*352'72) that if c is a serial family, T,= U CG'TK. Hence when K is a serial family, a vector which is not a member of C'TK may be the limit of a set of members of C'T, with respect to U,. It is the existence of an independent series UK, not generated by measurement, which makes the application of irrationals as measures possible. The following phraseology may be found convenient. Taking a unit T in a family ei, and an origin a in its field, if X e C'H' and S=X X K'T and = S'a = (X K'T)'a, we call X the "rational measure" of S and the 'rational coordinate" of x. We have, in the same circumstances, S= K 1 AT'X. x = AaS= Aa'C 1 AT'X.

SECTION C] MEASUREMENT BY REAL NUMBERS 443 We will call S the vector of X, and x the point of X; and the same phraseology will be employed for the vectors and points obtained by measures which are real numbers. We may now state the principle according to which we apply real numbers as measures as follows. Given a segment & of H, take all the vectors of a's: these form the class K n AT"I. Then the real number s' is to be the measure of the limit (with respect to U,) of the class Kc n AT T". Since U, has the opposite sense to that of TK, i.e. U, proceeds from the vectors with bigger measures to those with smaller ones, the limit we shall have to take will be the lower limit with respect to UK. Thus the vector whose measure is s': will be prec(UK)'(K AT") Now if we put X=s ', AT" = X'T, and X is a relational real number. Hence using *206*131, the vector whose measure is X is prec(U,)'X'T. Hence if "XK'T" represents the vector whose measure is X (unit T), we put *356 01. XK= prec(UK) | X c)f Assuming now that K is a serial submultipliable family, in which we take R as the unit and a as the origin, and putting, for notational convenience, P = U.Q=, we have first a set of preliminary propositions (*356'1 —191), of which the most important are II = (C'H') 1 AR;P=(C'H) 1 AR;Aa;Q (*356-13), P C'R, = 1AR;H' (*356'14), giving the relations between the series of ratios, the series of their vectors, and the series of their points. We proceed next (*356'2 —26) to the proof that X,,r -- 1. This requires, in addition to our previous hypothesis, that Q should be semiDedekindian. With this hypothesis, we first prove that if X, Y are relational real numbers, ('X, = P'Y, = Ka: X, = Y,.. X = Y (*356'21). We then prove, by the help of some arithmetical lemmas, that the lower limit of the submultiples of a given vector is the zero vector, i.e. tlp'SS {S K: (av). R = S} = I r C'Q (*356-22). Hence we easily prove that, if R is any non-zero vector, and X is a class of vectors having a lower limit L, the lower limit of the relative products of R and members of X is the relative product of R and L, i.e. X C K. =tlp'X. R eKa.. R I L tlp'R I (' (*356'221).

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 Remembering that the relative product is represented arithmetically by the sum, we may express the above proposition by saying that the limit of the sums of a given vector and a set of vectors is the sum of the given vector and the limit of the set. From this proposition we easily deduce that if RPS, X,'R + X,,S, whence it follows that X, r K e 1 -, 1 (*356-26). Our next set of propositions (*356'3 —33) is concerned in connecting the relative product of X, and Y, with the arithmetical product X x, Y, where " xr" has the meaning defined in *314. Here we only require that K should be serial and submultipliable, and we obtain X, I Y, = (X x, Y), (*356-33). This proposition is the analogue of *351'31 (except that K, is replaced by K); it has a similar importance, and calls for similar remarks. Our next set of propositions (*356'4 —43) is concerned in proving that the limit of the points of a segment of ratios is the point of their limit, in other words, that the limit of a set of points whose coordinates are a segment of rationals is the point whose coordinate is the limit of the segment. Here we again require that our family should be semi-Dedekindian; then if e is a segment of ratios, and X = s.', the above proposition is (X,'R)'a = seqQ'Aa"AR" = seqQ'Aa^X'R (*356-43). Here XK'R is the vector of X, (X,'R)'a is the point of X; AR"~ =X'R, and each is the class of vectors of members of ~; and A,"AR"4 or Aa"X'R is the class of points of members of $. Moreover X is a relational real number. Thus the above proposition states that the point of X is the segment (i.e. the limit) of the points of the ratios contained in X; i.e. of the ratios which may be considered less than X. We next proceed (*3565 —'54) to connect the relative multiplication of vectors with the addition of their measures. Here we require that K should be semi-Dedekindian as well as serial and submultipliable. We then find that if X, Y are relational real numbers, and R is a non-zero vector, (X,'R) I (Y,'R)

= (X +r Y),'R (*356'54). This proposition is the analogue of *351'43, and calls for similar remarks. The proof proceeds without much difficulty by means of *356'43. Finally we have a set of propositions (*356'6 —63) to prove that the real number which measures a rational vector is the real number corresponding to the ratio which is its measure; i.e. if X is a ratio, the vector which has the ratio X to the unit has the real number s'H'X for its measure. It is to be remembered that rational real numbers must not be identified with ratios,

SECTION C] MEASUREMENT BY REAL NUMBERS 445 any more than integral ratios (i.e. ratios of the form v/I) must be identified with cardinals. The real number corresponding to a ratio X is 5i'H'X; this is what we call a "rational real number." In measurement, when we are measuring by ratios, if R is our unit, X will be the measure of X C Ic'R; but when we are measuring by real numbers, the measure of X ~C c'R must be a real number. The real number which is the measure of X ~ ic'R will, by our definition, be a real number Z such that X CR = prec (U,)'Z'R. Thus we have to prove that, if X is a ratio, the above equation is satisfied if we put Z=s'H'X. This requires that c should be serial, submultipliable and semi-Dedekindian; we then have X E C'H. d. ('H'X), = X K (*356-63). -) Thus although the "pure" real number s'H'X is not identical with the "pure" ratio X, yet the "applied " real number (S'HJX)K is identical with the " applied " ratio X t c. This fact explains why the results of the habitual confusion between a ratio and a rational real number have not been even more disastrous. *356.01. X, = prec (U) X rK Df *356'1.:. Rec.): S=X,'R.. S=prec (UK)'S'R [(*356'01)] *356-11.:. R e K: S=(S R. =(.- S = prec (U)'AR"A [*356'1. *336'12] *356'12. F:. K e FM sr subm. X, Ye GH'. R e Ka. a e s'(U K. Q = 'Ka. P = U.: XH' Y. (X C K'R) P (Y 'R). =. {(X C K'R)'a} Q {(Y K'R)'a} [*352-73. *336 4] *356-13. F: K e FM srsubm. Re ic. a e s'("K. Q = 'ca. P = UK.). H' = (C'H') 1 AR;P=(CIH') 1 AR;Aa;Q [*356-12] *356-14. F: Hp *35613.. P C'R, =K AR;H' [*352-72] *35615. F: Hp *356-13. X C C'H. X = s'.. maxp'X'R = /c 1 AR"maxH' Dem. F. *352-41.: Hp.. X'R C'R. X'R = AR" (1). (1). *35614. D F: Hp. a. maxp'XR = max (PC C'RK)'X'R [*356-14] = c 1 AR"maxH'X:) F. Prop

446 QUANTITY [PART VI *356-16..-F: Hp *356-13. X E.X = ~'X.). maxp'X'R =A [*356-15] *356-17. -: Hp *356-16.).K= Itp Xr C'P [*356-16] *356-18. -F: IceFMconnex.:).XKEI —+Cls [*206-161. *:336-62. (*353-01)] *356-19. F:.K eFM sr.P= UK.)ZeO'IH..Z K;P CP Dem. F. *336-511.) F:. Hp. R, e K/z, ve NC ind - t'O. Z =,ulv.) BPS.. RAsPSA. [*350-43]):RPS. M = (p/Iv) ~ K c'R. N = (pa/v) ~,c'S. MvPNV [*336-511] D. MPN:.)F. Prop *356-191. F: Hp*356-19. Xe~i"C'i.:). X ~ cP CPIX~ K Dern. F. *356-19. D F:.Hp.:):XeC'e.X=~i'X.ZeX,.).Z~/K PC-PIZ~K:.:)F. Prop *356-2. F: Hp *356-16. E'e0. L eX -,at.:).KcA.R'L ep'P"A.R"C Dem. [*206'6. *352-12]:). KJ1AR'L ep'Kc1 AR;H"AR",L4. [*356-14] D. Kc1 AR'L e p'(P" AR"ccp:

D F. Prop *356-21. F:. K e FM sr subm. C MvY~Ka e semi Ded. X,Ye h""G.): (1'XK, = (I C= Ka K = Y,. X = Dern. F. *356-16.*214-7.) F: Hp.X ~ C'e X - 'X. Y= 'IL.RE Ka.:). E! Xc'R. E! Y,,'R (1) F.(1).*35562.:)F:Hp(l).P=Uh.H!X-u.:).(YK,'B)P (XK'B) (2) Similarly F: Hp (1). P= U, 21! p - X.D. (XW'(R) P(Y,'R) (3) [Hp] D.X=Y (4) F. (1).(4).) F. Prop *356-211. F:o-,TrNCind-t'O.vENCind-t'O-t'1l.). (o+0.r)V > U", +,0 (v XG av-cl X,,r Dem. F.*1 13-43,66. *116-34.) F.(a-X, r)2 = &2+0(2 X0o X T)+0r2 (1) F. *1 26-5 D) F:. Hp.)(o- ~0> 0", +0 (v x0, o-,cl X0 Tr). (0r +0 T) V'+cl > Ov01 "+0 (V X0 O"V X0 T) +0 (&' x X7) (2) F. (1). (2).Induct.:)F. Prop

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SECTION 'C] MEASUREMENT BY REAL NUMBERS 447 *356-212. F:p > a8P)a, ~e NC ind..'a(Hp).v eNC ind PI,> arvXc Dem. F. *356-211.)D F. (1).*1 26351.')F:Hp (1).aT~,(V X0) >o0-Xc.).pv>ovXc (2) F. (2) *1134:3. *120-416. *126-a-.) F: Hp (1). V X,,'iT> a- x0 1).)D. pV> a-v x0 D F. Prop *356-213. Fp > r. p, a-,,e NC ind. q 0. D (v v 6 NC itid. pP' x0 ij > a," x,,4 Demn. F. *356-212.) F p.). (av). v eN C nd. pV,> a v x0~~ Prop *356-214. F:p, a e NCind -t'0. p>a. XEGC'H.). (v.v 6 NC inid. (p/oj"HX [*356-213] *356-215. F:XeC'e.'p,a-eNCind-t0'0.p>r.:). (ax). X E X. X X, /-' Dernt. F.*305-142. Induct.) F: C G'1H. a!v X.p NC ind - tO X 6X. DX. X X, P/a -X:) X XeX.)X. X x~sp"/&',EX [*356-214] D:H"X =G'H(1 F. (1). Transp.) F. Prop *356-22. F: Hp*356-13.Qeserni Ded. D. tlp's{IseK:(av) R-&} = -Ij'GQ Dent. Se,Ec.SILxc-,= R.).-L",PS": [*:301-5]): Te c. T& = R)6. L"PT: [Hp] D:L",P*L(1 F. (1).(2).D)F:HP.D). L ',,6 ~c:D)F.Prop *356.221. FHip *356'19. Q-~K. X~ ~C.KL tlp'X. Re ica.). R L =tlp'R "A Demn. F. *334i15.*336-411.:)F:. Hp.D: LPM.D. (RjL)P (R M): [*37'61]): R"X CP'l(R iL)(1

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448 QUANTITY [PART VI I-.*336r41.) F: Hp. (R IL) PM.:). (gjN). Ne Kca. M = R I L N. [*330-31])(aN).AIVe Ka. R IM =L N. [*336-41.*334-1.3]).LP (RI M). R I Me Ica. (2) [HP]). (aN). N XNP (R IM). [*37-1].Me6P"cR I IX (3) F-.(1). (3). *207-21.:)F. Prop *356-23. 1-:Hp*3'6-22.RPS.:).(av).veNCind-t'O.[I(v+01l)lv ~ x'R]PS Dem. F. *356-22-221 F)I Hp. X = T{ITE K:(av).- R = ToJ.) tlp'R I"cx= R. [HP]) (3v).vTeNX.(RIndT'O P R(1S)K'} [HP0].*332).(dv).- v e NC ind - t'O. t-v +I l)/v} ~ KIc'] PS. -.Po *356-231. 1- Hp *356-23.).(ajv). v e NC ind - t'O. SP [{(v -c, 1)/ vi ~Kc'R] [Proof as in *356-23] *356-24. F: Hp *356-23.X e "IC'O.).)Xc'R fX,'S Demn. F.*356-23.) F:Hp.XVECIO X-= 'x. (gjp, o-). po a NC ind - t'O. p>o-. {(p/ o-) ~ K'RJ PS. [*356-215]:. (Hpo,oY).p, o-ENCinduct-t/O. p>u. YeX. Yx,,p/-,EX. J (/- KIcR} PS. [*336-511]:).(ap,cr.,Y).p,u-eNCindI-t'O.p>-.Ye'X.Yx~p/o-ep'H"Xk. I {Y K'(p/O-) ~ K'BJ P { Y Kc'S} [*356-1] D. XW'"R YK'IR: F. Prop *356-25. F:HP *3'6-22. X eh"C'E.). XgR C Q Dem. F.*356K121. F:Hp)X,, RE Ka (1) F. (1).*41 13. DF..Prop *356-26. F: Hp *356-25.). X,,rIe-.+1 Dem. F.*356-24.Transp.:)F: Hp.RSEKa.X,c'R=XKc'S:).-R=S (1) F. (1). *35618-2I.) F. Prop

SECTION C] SECTION c]MEASUREMENT BY REAL NUMBERS49 449 *356-3. F:. Ke FM apconx subm. s'Pot'%'c C K. ~, v e C'60. R, Se K.)D: R (~~'F XrS'6v) AS..R {(cF IK (~'V)}AS Dern. ~~~~~ (1)~~~~~~~~~~~~~3 I-()) F:. Hp.):] (s'jz xrS'~v) S..(3M, N). Me/ t.LN ev. R(Mx, N) S. [*3-1-31P22] =E.(aM, N). M ep.tNe v. R (M rK N) S:.:)F. Prop *356-31. F: E FM ap conx subm. sPot",cC c. X, Ycs"GC'e.) *356-32. F:KicEFMsrsubm.X, Yeih"GC'O.R?,ca.).XK' yK'"R=<XYKR Dern. F.*356-191.)F:. Hp.): S K r% YR. D. Kn X'S CP"CX'YK'R: [*37-63]): X"C'(r Y'R) C P"~XCY,,R (1) F.*305-6.)F:Hp.XECG'eX=s'X.ZZ'eX.ZHZ'.D Z~ K C'YK6R =Z' ~ tc'(Z IZ') ~ K" Y/ R. [*356-17]).Z ~ K6Y1TR e Z ~ K"PT"Y'R. [*356-19] D. Z ~ K6 YKC1 E P"Z' K(Yc"YR. [Hp]).Z ~ K' Yc'R e P"CX" Y'1 (2) F.(1).(2).)F:llp.). PlX"CY'R=P"X'YK'R. [*356-1]:). (XI YXK'R = Xc6'Yc'1: D F. Prop *356-33. F: Hp*356-32.)-XJfK =(X Xry)ic [*356-3132] *356-4. F:KeFMconx.Q=CnV'~'Ka~. SEK.aCC'Q.H!ax.E!seqQ'a.:). S'seqQca = seqQCSCa Dem. F. *330.563. D F: Hp.:). S seqQ Ca Ep'CQ"CS~a (1) F.-*37-1. D)F:: Hp.)3:. S'zeQC9p'Q"~SCa.: (ay): x e a.)D,,. S'xQy: yQS'z: [*330-542]:(aw): xWE a.)x. S'x Q S'w: S'w Q S'z: [*208&2]: (aw): wca.)D, xQw: wQz: [*37-1] z c Q"p'Q"a (2) F.(2). Trarisp.)F:.Hlp.):zc., cQ"p'Q"'2.E.S'zr,..'Q"'p'Q"ISia (3) F. (1). (3). *330-542.)DF. Prop R. & W. Ill. 29

450 QUANTITY [PART VI *356-41. F:./xEFMconxtrs.P= U,.Q=9"'Ka.aEG'Q.XCK.H! X.:): N = seqp'X. E.N E K. seqQ'A,,"X = N'a Dem. F. *336-43-2.*206-61.) I-:. Hp.)::N =seqp'X. NeK t. Aa'N= seq (Q ~ Aa"'K) 'Aa"X (1) F. *206,21 1.) I- Hp. b =seqQ'A,. 'X.. (aR). 1? e X. R'aQb. F. (3). *336-11.) F. Prop *356-42. F:Hp *356-41. E!seqp'X.).(seqp'X)'a =seqQ'Aa"X [*3056-41] *356-43. F:Hp*356-22.~,EC'e. X=hl.aeC'Q.:). (X,-~a= seqQ14,ARA~ = seqQ'A/'(X'R [*3-5642-11-21. *336-12] *356-5. F: Hp *356-22. X,? Ye 9'P'1. aEG'CIQ. REKi. XK A X'R.1&= KA IY'R.). -4I - 4+ (X"'R)'(YK,'1)'a = seqQ'8'X'seqQ 'h'euta Dern. F. *356-43. *336-12.)D F: Hp. (X/,R)I(Y,,"'R)'a = seqQsX'(YK'BR)' [*356-43.*336-12] seqQ'YX'seqQ'l'itt'a: Prop *356-51. F: HP*35D65.).(X +rY)'R = seqp's'X ",tt Dem. (X +r Y),,'1 = seqp'A. R"(~ +a n) [*312-32.*31 1 11.*308-32] = seqp'A R"ls'f +St [*336-11] = seqp'V t (aL, M). L cE.ME eq. N = (L +, M) ~ K'R} [*351P43] = seqp'N I (:fL, M). L E ~. MEfl. N = (L ~ Ic']?) I(M~ K 'R)} [HP] =seqp'Nt(aU,W).UE'X.WE/.4.N=UjW}::)F. Prop *356-52. F:HP *3565.). t(X +rY),~'R}'a =seqQ'(g,'X)"91t,'a Dem. F. *356-51.) F: Hp.) { (X +.Y).,'1?}"a = (seqp's'X I"At)'a [*356-42] = seqQ'Aa"S'X)I "t [*336-11] = seqQ',~ {(aX, Y).- XeX. YE-. x = (X IY)'a} [*41-11] -=seqQ': {(RX). X eX. x EXllhl" Ittac [*41-11] = seqQ'(hA'X)"i'h~'pa:D)F.Prop

SECTION C] SECTION a]MEASUREMENT BY REAL NUMBERS41 451 -4+ -4) -4 *356-53. F:HP *3.56-5.DseqQ8hXlseqQ8h6ta -seqQ6(SM)6Y- C/a Dern. F.*356-

16..) F: Hp.). seqQ C 6.seqQs/lp a = tQYX'seqQYs/ua [*356-4] =ItQ'X" {(3L). L E X. x = seqQ'L"'s/' kal -4+ [*207-55] = ItQ~s CA {(aL). L eX a= = ci [*41P11] = ltQc(SCXY YLaa [*356-16] = seqQ'(h6'X)"Y C~a:) F. Prop *356-54. F: Kc FMsr subm.Cnv'~'Ka Csemi Ded.X,YE,"G'CE).REKa.D. (XK'1?)I (YKc'1?) = (X +, Y)KI'R [*356-553-52] *356-6. F:KEFMsr.REKa.P= Uc.Q'hKa.X6C'H.) K n AR"CH'X CP'X ~K"R Dern. F. *37-6.)F:. Hp.): M eA.R"II'X..(3[Y). YHX.MYR. [*352-7].MHP (X ~K`R):.:)F. Prop *356-61. F:Hp*356-6.KEFMsubm.QesemiDed.SP(X~K'R).:). (Y). YHX.SP (Y ~lK'R) Demn. F.*35'6231.)F:Hp.). (dv). v CNC md - '.SP [{(z' -c 1)/ v} / K'X / 'R] [*351-31]:). (dp). ve NC ind - t'O. SP [{(v -, I1)v x. XI ~ KCR] [*305-7151] D.(UY).YHX.SP(Y~K'R):DF.Prop *356-62. F:Hp *356-6.K eFM subm. Q Csemi Ded.:). PCX ~ K'-R C P"CAR"CH'X [*356-61] *356-63. F: Hp*356-62.). (~H'X)K=X ~K Dern. F.*356-662.) F: Hp.). X'~tK'R = tp'AR"cH'X. F. (1). *356'21..)F.Prop 2 9- 2

*359. EXISTENCE-THEOREMS FOR VECTOR-FAMILIES. Summary of *359. In this number we prove that, assuming the axiom of infinity, there are vector-families of the various kinds considered in previous numbers. If P is any well-ordered series having no last term, the converses of the interval-relations, i.e. the class finid'P, form an open family of CP (*359'11). If P is a progression, this family is serial and initial (*359'12). The family consisting of additions of positive ratios to positive ratios (including Oq), i.e. consisting of all terms of the form (+,X) CH', where X e C'H', is initial, serial, open, and submultipliable (*359'21), assuming the axiom of infinity. The family consisting of generalized additions of positive ratios to generalized ratios is serial, open, and subrultipliable, but not initial (*359-25). The family consisting of multiplications of positive ratios not O0 by positive ratios not Oq is open and connected, but not serial or submultipliable (*359'22); if we confine the multipliers to ratios not less than 1/1, the family becomes serial (*359-25). The family consisting of additions of positive real numbers to positive real numbers (including '0Oq) is serial, initial, and submultipliable (*359'31); the family consisting of generalized additions of positive real numbers (including 'Onq) to generalized real numbers is serial and submultipliable, but not initial (3359'32). Similar propositions hold for multiplication, provided t'Oq is omitted; but the resulting families will not be serial. In the case where the field is confined to positive real numbers, however, the family becomes serial if the multipliers are confined to such as are not less than H'(l/l), which is the real number 1. The last set of propositions in this number (*359'4 —44) are concerned in proving that, given a family c whose field is 83, if S is a correlator of a and /, Sti"c is a family whose field is a, and which has the same properties of being connected, open, etc. as the original family Ic. Hence if c is a family whose field is the real numbers, and we are given any class a similar to the real numbers (in other words the field of any continuous series), if S is the correlator

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SECTION C] EXISTENCE-THEOREMS FOR VECTOR-FAMILIES 453 of this class with the real numbers, St"K/ gives a family whose field is a. Hence from our previous existence-theorems we derive the existence, for a, of an initial serial family, giving us a system of measurement for a. Similarly if a is similar to the rationals. *359-1. F: P e f~.2 E! B'P.). finid'P e Cl ex'cr'C'P Dem. F.*260-23-28.) F: Hp.)finid'P C 1 — +1 (1) F.*121-302.)I-:Hp.).D'1P0=C'P (2) I-.(2). *121 302-35. *260-28.) F: Hp. v e NC ind. D'P, = C'P.).D'P,+,i= C'P (3) F.(2). (3). Induct.) I-: Hp. Re finid'P -. DR =C'P (4) F. *121-322.) F: Re finid'P.).UI'R C C'P (5) F.(1). (4).(5). *330K1. F. Prop *359-11. F:PefL. E!B'P.).finid'Peffmap'O'P Dem. F. *260-28. *121:352.:) F: Hp.. finid'P EAbel (1) -.*71-19. F: Hp.u, v e NC ind. f! P,, JP"J../4v (2) F. *121-35. D F: Hp (2). ~u >v. D. pi I PY C PfA-cV [*91 6.* 121836].(PI P')P0 C J (3) Sitnilarly F Hp(2).v(P, II)POCJ (4) F. (2). (3). (4).:) Hp. L e(finid'p),a Lp~. G,J (5) F.(1).(5). *3591.)F.Prop *35912. F:PEW.K=finid'P.).Ke fmsrinit'C'P.S'Ka=P Dern. F.*263-14-141.*1221.) F: Hp.). c'K'B'P== C'P (1) F. *26314-141. D F: Hp.)D. 'K = P. (2) [*334-31.*3.5911] D. K E M sr (3) F.(L).(2). (3). *335-14.)F. Prop *359-2. F: Infin ax. c K=iR {(qX). Xe C'H'.I =(+8X) C'H'.). Kce FM.S'Ia -H1 Dem. F. *306,54-25. *304A94. F: Hp.). KCI -+ 1 (1) 1. *306-25. *304-49. F-: Hp. Re K.:.).PR = OR'l. D'R C C'H' (2) F. *306-11-31. D1: Hp.R, Se / c. R RIS=SJR (3) F-.*306-52. DF: Hp. D. 9'Ka =H' (4) F. (1).(2). (3).(4).DF. Prop

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4.54(QU~ANTITY [PART VI *359-21. F Hp *35992.. EFM init sr subm. ~a= H' Dent. F.- *306'24. F: Hp.)D. BCK = C'H' (1) F.*306-41. D F:. Hp. Xe C'H'. tk v c NC ind - t'O. S= {~, (X xS I/v)j CH'. D: SI = 1fs (X x.,pIv)I C C"H'.. S -XIL+ = 1 +, (X x, p -,,IIIv)I C CH': [Induct]): S= f+,(X x,,a/v)} j G'H'[*305 51]): = (+,3 X) ~ C'H' (2) F. (2). *3511. *359-2. D)F: Hp. D. K e FM submn (3) F.(1).(3). *359-2. *334-31.)DF. Prop *359-22. F:Infinax.K =1?{(2X).XeC'H'.R=(+gX) C'Hg}.). Ke FM sr subm. -HIca The proof proceeds as in *359-21, but in this case there is no origin. Every member- of Kc is a connected point, i.e. a member of conx',c. This results from *308-54. If, in *359-21, we substitute H for H', the proposition holds except that Kc has no origin. *359-23. F: Infinax. K=R {(,1X).XeG'H.R==(xX) C'H}.). K EFM ap conx The proof proceeds as in *359-21. We have to take H instead of H', because (x, Oq) C C'H' is not 1 -* 1. We do not get K E FM subm, because not every rational has a rational vth root. *359-24. F: Infin ax. K = R {(aX). XE CG'1 - ff(qj. R =(x9 X) C ('Hg - t'Oq).. Ke CFM ap conx The proof proceeds as in *359-23. *359-25. F:Infinax.Kc=kI(HX).(1/1)H X. R=(xX) C'H}.). Kc e FM sr. "Ka= H The proof proceeds as in *359-21. *359-31. F:Infin ax.K=R = I ((gp). ~eC CO'.R= =(+p pu)CC CO'1.). Ke FM sr init subm. "K5 =a Dem. F. c*311P74. DF:Hp.D.KC1-* (1) F. -*311 27. D)F: Hp. Re Kc.).tI'R = CO'. D'RCC'C' (2) F-.*311243 1. DF: Hp.)D. KeObel CIO, = initl/c (3) F. 311-12-121. D F: Hp. D. K Abel 4 F s *311-65j. D F: Hp. D. k,, I a (5)

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SECTION C] EXISTENCE-THEOREMS FOR VECTOR-FAMILIES45 455 F.(1).2).(3. (4).5).)-:Hp).KE~srint?hc~(9~' (6) F(6).-*310-1 51.*351 11.-Hp)Ke FM subrn (7) F(6). (7).) F. Prop *359-32. F: Infin ax. K=R{[(Eae6CO' 'R = (+a,.L) ~C'E~g1.) Kc FM srsubm. 8'/C= 0 The proof proceeds as in *3059-22. Similarly the analogues of *359-23-242.5 can be proved for real numbers; the resulting families, in these cases, will be submultipliable, but it will be necessary to omit L'Oq from their fields. *359-4. F: cE Cl ex'cr', SEc ai 9-*:). St"Ke Clex'cr'ca Dem. F. *330-1. *71252.)FHp.:). St"K C 1-+1 (1 F.*150-21211. *330-1.) F:Hp. RE6 St""K. W R = S",8. D'RR C CU'R. [*73-03].PR =a. D'R Ca (2) F.(1).(2). *330-1.) DF. Prop *359-401. F: KEAbel.SeCls-*1.sK~L",cCWIS.).St",/ceAbel Dern. F.*72-601.:)F:.Hp.:):P,QEc.:).PISIS=P.QI~SISQ. (1) [*33005]= IQiPI [(1).*150-1] = (St Q)I (St P) (2) F.(2). *330-5. F. Prop *359-41. F: KC EfM'/. S6a srn/38.:). St"K Efm'ta [*359-4401. *330'51] *359-411. F:K EFM.a econx'K.S El — 1. S'fII"K = (IPS..S'a Econxst'St"K Dem. F.*15l11.:)F:Hp.P-S;'K.:).SEPsilors') [*15L33]). ~~P'S'a u P'S'a =S".YK'a v tgKa [*331 1] -Sc S I"sPK [*330-13.*150-21 1 = P'S; 9K [Hp] =-GPP1 F.4*50-16.)F: Hp (I) -..P = 'St"K (2) F. (I).(2) *331-1.)IF. Prop *359-412. F:K Efm conx'J3. S6a sni/3.). St"K,cEfM conX'a [*359-41P411]

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456 QUANTITY [PART VI *359-413. F: cEFM ap. SeI — I.S'P'66K=U'S.).St"KeFM ap Dern. [*200-21])S (P IQ)PO CJ. [*150-83] i.s; (P IQ)}PO cJ (2).*3-9-4.:)IF: Hp.:).St"Ice FM (4).(:3y.(4).-*333*101.) F. Prop *359-414. 1-: ceFM.Sel1-?1. s'"xIlt=W[S.a~init'ic.).S'a=init'St",c [Proof as in *359-411] *359-415. F:, EFMsubm.SeI-+I. 1Ss('I.).StI"xEFMsubm Dem. F. *301-21.)F: Hp. YEx.ve NCind.) YV,+cl = YVIY (1) F. (1). *72-601.)DF: Hp. S;Yv -(S;Y)v.:). S;Yv+c = (S;Y)P,+c (2) F. *351-1.)F:Hp. veNC ind - t'0.X cK.).(Y). X =Y". YeKt. [(3)]:).(afY).YEK.S;X= (S;Y)v (4) F.(4). *351 1. *359A41.)IF. Prop *359-42. F: [! fmconx apsubm'fl. acsm 3.).g~! fin conx ap subm'2 [*359-41-412-413-415] *359-43. F:Peit D)a FM init srsubm(^'kc ~a= P) [*359-4221-414. *274-44. *123-18. *304-47. *273-4] *359-44. F: Nr'P4-i9. a! FM init srsubm n ',c (6-a=P) [*35942-31'414. *275-3. *310-15. *204-47]

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SECTION D. CYCLIC FAMILIES. Summary of Section D. The theory of measurement hitherto developed has been only applicable to open families. But in order to be able to deal with such cases as the angles at a point, or the elliptic straight line, we require a theory of measurement applicable to families which are not open. This theory is given briefly in the present Section. When a family is not open, two vectors which have one ratio will usually also have many others, i.e. we shall not have 3!X i c A YC.KD X = Y, where X, Y are ratios. Also a ratio confined to the family will not usually be one-one. Under these circumstances, it is necessary, if measurement is to be possible, that there should be some way of distinguishing one among the ratios of two vectors as their " principal" ratio, and of then showing that, by confining ourselves to principal ratios, the requisite properties of ratios reappear. The case of angles will serve to illustrate our procedure. Considered geometrically, not kinematically, a vector which is a multiple of 27r is identical with the null-vector, and if 0 is any angle, 0 = 2v7r + 0, where v is any integer positive or negative. We are here considering an angle as a vector whose field is all the rays in a given plane through a given point. Thus there will be two angles which are half of the null-vector, namely wr and 27r, and four angles which are a quarter of the null-vector, namely 7r/2, r-, 37r/2 and 27r; and so on. The ratio of 7r/2 to 7r is any number of the form (2p + 1)/(4v + 2); thus two terms may have many different ratios. In order to evade this difficulty, we first arrange angles in a series ending with 27r, and having no first term, but proceeding from smaller to greater angles. Then the angles which have a given ratio pf/v to a given angle will be finite in number, and therefore one of them will be the smallest. We take this as the "principal'" angle having the ratio t/v to the given angle, and define "(t/v),," to mean the relation between two angles consisting in the fact that the first is the " principal" angle having the ratio 1/Il to the second. Then of all the ratios between the two angles, the ratio p/v may be regarded

 as the "principal" ratio. It will be found that, with suitable hypotheses, (pU/v), has the properties required in order to make measurement possible. In order to make the above method feasible, certain properties must be assumed to hold concerning the family K. (These properties are all verified in the cases that arise in practice.) We shall therefore only speak of a family as cyclic when it fulfils the following conditions: (1) It must be connected. (2) It must contain a nion-zero member which is identical with its converse. This is the property which makes the family cyclic. In the case of angles, the member in question is 7r. (3) It must be such that ca 1 U, is transitive. This is the property which enables us to arrange the field in a series. It will be observed that U, cannot be transitive, since, if K, is the member which is its own converse, we have (I s'(CP'K) UK K,. KK U, (I s'c(["), but we do not have (I r S'(I"K) UK (I r s'aI"K), because UK is contained in diversity (by *336'6). It is, however, possible that UK should be transitive so long as we do not start from I s'C("K, and this we assume as part of the definition of cyclic families. (4) In order to avoid trivial exceptions, we assume that K does not have only two members, since otherwise it might consist only of I [rs'I"c and KK. We are thus led to the following definition: FM cycl = (FMconx - 2) n K {K 1 UK e trans: (aK). K e a. K = K Df. We prove that there is only one such relation as K, and therefore put K, =(K) (Ke. K-K) Df. Also for the sake of brevity we put I, =I Sa"K/ Df. We then prove that K is a family having connexity, and satisfying the condition I)D"cK = aclK, i.e. having the domain of a member always identical with the common converse domain. Thus by *334'21, K,= K v Cnv"K. In a cyclic family, K v Cnv"Kc consists of two mutually exclusive parts, namely Ka and K,, "Ka. (In the case of angles, KIR would be + i+R. Thus Kca would be the angles from 0 (exclusive) to wr (inclusive), and K, "ca would be the angles from 7r (exclusive) to 27r (inclusive).)

Also K,l "'cK consists of the converses of c- tKK,. We take up next (*371) the question of arranging K v Cnv"K in a series. For this purpose, in order to avoid circularity, we have to erect a barrier at some point; we choose 1, as this point. By the definition of cyclic families,

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SECTION D] CYCLIC FAMILIES 459 Kc 1 U, is transitive; hence, since the family has connexity, U, fca is serial. This relation therefore arranges all the members of Ka in a series, beginning with KK and proceeding towards I,. In order to extend our series to K,, 'Ka, we only have to make K,R precede KI S if R precedes S, where R and S are members of ca. That is, we arrange K,, "ca in the order KK I; UK i Kc. This gives a series which begins with I, and proceeds towards KK without reaching it. Thus taking the sum of the above two series (in the sense of *160), we get a series whose field is K vCnv"K, which Wbegins with 1K, travels through K, I "Ea to KK, 4I, and on through Ka towards I1, without quite reaching I1 again. This relation we call WK; the definition is WK = KK; UK at- U, Df. KA, I Ka Taking an arbitrary origin, a vector may be indicated by the poirnt to which it carries the origin. Thus in the figure, I, is at the origin, K, is opposite the origin; the upper semi-circle, including both ends, is K; not including the right-hand end, it is Ica; the lower semi-circle, including both ends, is Cnv"/K; including K, but not I,, it is Cnv"'ca; including 1K but not KK, it is KKI "K. Then WK starts from IK, and proceeds through the lower semi-circle first, and afterwards through the upper semi-circle, stopping just short of 1,. If K is cyclic, WK is a series. Under most circumstances, if ReK, we shall have PWQ.).(PIR) W,(QIR). The investigation of the various cases in which this holds occupies a large part of *371. In the remainder of this Section, our work becomes more full of ordinary arithmetic than it has been hitherto. We shall therefore, where cardinals are concerned, abandon the explicit notation we have hitherto employed, and substitute the ordinary notation. Thus we shall write 4 + v in place of, +c v, and rL in place of / xo v. We shall, however, retain a -o v for subtraction, in order to avoid confusion with the sign of negation of a class. We proceed next (*372) to consider what is in effect the class of vectors not greater than the vth part of a complete revolution (e.g. in the case of angles, not greater than 27r/v). We define this by means of the relation WK. It will be seen from the figure that if R is a non-zero vector, we shall have Ra+I W,R'R, unless Re belongs to the lower semi-circle and R-+1 to the upper, in which case RW WK,R+I. The first time this happens is the first time that R+1 becomes greater than one complete revolution. Hence if, for every number a less than v and not zero, R~+I WCKR, it follows that RV is not greater

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460 QUANTITY [PART VI than one complete revolution, and therefore R is not greater than the vth part of a complete revolution. The class of such relations we call v,; thus we put v, = (K Cnv"Kc)n R (a < v. '4 0. 3,. R++1 WKR) Df. The main

propositions to be proved in this subject are P e v. P W Q.. P" WKQ and (what is an immediate consequence) P, Qe v,.:P^=Q.=.P=Q. This latter proposition is the foundation of the theory of principal ratios. Another important property of VK is WKC VK C vK, so that V, is an upper section of WK. We proceed next (.373) to consider submultiples of identity, i.e. vectors R such that RV=IK, where v is a cardinal. We assume here, and almost always henceforth, that K is a submultipliable family. We first consider vectors which can be reached from KI by successive bisections. We know that KK2= IK; if R2= KK, then R KK, because KK2 = KK. Hence by continuing the same process we arrive at the existence of a vector Q such that Q2V=I,:p < 2V.p+O.)P. QoPI,,. Hence we easily arrive at the result that, if v is any inductive cardinal, there is a non-zero vector whose vth power is 1K. (This does not follow from KceFMsubm alone, because IK = I, so that from the definition of FMsubm we cannot know that there is any vector except IK whose vth power is IK.) Thence we prove that there are non-zero vectors whose vth power is IK, and which are such that no earlier power is IK, i.e. we prove (M3R): R ca Rv=I,: a- <. Ra-o <.. R' I,. The class of such vectors we call ('K, v). If R is such a vector, the number of different vectors which are powers of R is v. Hence the powers of R have a maximum in the order WK; since W, proceeds from greater to smaller vectors, this will be the smallest vector, other than II, which is a power of R. Concerning this vector, we show that it is a member of IK, i.e. it is such that, if a- < v. oa + 0, R'+1 WKR". Finally we prove that there is only one member of v, whose vth power is I,. This will be what we may call the " principal" vth submultiple of IK; in the case of angles, it will be the angle 27r/v. It will be observed that 27r,/v always has identity for its vth power, and has no lower power equal to identity if u is prime to v. Thus the uniqueness of the " principal" vth submultiple depends upon the fact that it is a member of vK, so that, by what has been proved in the previous number, no other member of v, has the same rth power.

SECTION D] CYCLIC FAMILIES 461 We next, in a short number (*374), extend the last of the above results to any vector, proving that, if R is any member of C V (Cnv"c, there is a unique member of v, whose vth power is R. We may call this the "principal" vth submultiple of R. We prove also in this number that, if S is the principal vth submultiple of 1K, V, consists of all vectors not earlier than S in the order WK) i.e. of all vectors not greater than S. Finally (*375) we define "principal ratios" and show that they are oneone and mutually exclusive. We denote the " principal ratio " corresponding to ul/v by "(/V)." This is defined as the relation holding between R and S when the principal Itth submultiple of R is identical with the principal vth submultiple of S; that is, we put (frI)/v =RS{(RT). Te ^ n. R- =Th.S= } Df. It is obvious that (/v),K G (ft/v) t c,; and there is no difficulty in showing that principal ratios are one-one and mutually exclusive. We have not thought it necessary to carry the development of this subject any farther, since, from this point onwards, everything proceeds as in the case of open families. We have given proofs rather shortly in this Section, particularly in the case of purely arithmetical lemmas, of which the proofs are perfectly straightforward, but tedious if written out at length.

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*370. ELEMENTARY PROPERTIES OF CYCLIC FAMILIES. Summary of *370. In this number, after the definition of cyclic families already cited, we proceed first to prove that only one non-zero vector is equal to its converse (*370'23). This one we define as KK. Next we prove that, if R is a nonzero vector other than KK, RI K1 is the converse of a non-zero vector, and R1 K, is a non-zero vector (*370'31'311), whence it follows that D'R = IR s='(c"K (*370-32), whence further we obtain D"C = (C"K. K e FMconnex (*370-33). Hence further, since by definition Ka 1 UK is transitive, it follows that Ka 1 UK is a series (*370'37). The remaining propositions (*370'4 —44) are concerned with the relations of the two semi-circles Ka and KK "K (cf. figure, p. 459). We have Cnv"K = KK I " (*:370-4), K n Cnv"cK = t'I, u 'K,K (*370'42), K,K " = Cnv"/c - tK, (*37043), and Ka n KK " Ka = A (*370-44). *370-01. FM cycl= (FMconx-2) n {Ka1 UKe trans: (tK). K e K. K=K} Df *370'02. KK= (IK)(K e a.K=K) Df *370'03. IK, =I rs AK Df *370-1. F:./ce FMcycl.: e FM conx -2. Ka1 UU etrans:(aK). K e a. K=K [(*370'01)] *370-11. F: K e FMconx.. c 1 U., C J [*336'6. (*336'011)] *370'12.: ce FM conx. a1 U, e trans. R, S e a. RUKS. SUKT.). R T [*370-11] *370'13. ': K e FM. K e. K = K.. K = I, [*330-31]

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SECTION D] ELEMENTARY PROPERTIES OF CYCLIC FAMILIES 463 *370-2. F:. KeFMconx.KIca1 Ukcetrans. Ke/ca. K=K.:): 1?e Ka - R Kci C K.).1U,, (RIK). (RI K) U,,R Dem. F. *3 7013.)F Hp.:). R IK2 =R (1) F. *336A41. (1).)DF: Hp.). -R U,(R IK). (R IK) UKR:)DF. Prop *370-21. F:Hp*370-2.RE6Ka.1? KEK.).IK=IK Dem. F. *370O12.Transp.:)F:Hp.RU,,(RIK).(RIK)UKR.).RIKe-..EKa (1) F. (1). *370-2.:)F. Prop *370-22. F: Hp*370-2.ReK&-t'K.).1?K-.EK Dem. F.*370-21.*330-32-5. -: HP *370-21). R K (1) F. (1). Transp.) F. Prop *370-23. F:Hp*370-2.REKa.R=R.:). R=K` Dem. F. *331-33. F: Hp.:). RKe Kv CnV"K (1) F. *330-552.*34-2.:)F:Hp.:). RK= Cnv'(R IK) (2) F. (1). (2). F: Hp D..R K cKi. [*370-22.Transp] D. R = K: D F. Prop *370-24. F: KeFMcycl.).E!KK [*370-123. (*370-02)] *370-25. F:.K EFMcycl.): RE Ka.R =R.=r.R=K, [*.370-24.(*370-02)] *370-26. F: KEEM cyci.). K,,,Ka.KK=K.K KK2=hl [*370-2425-13] *370-3. F: Ke Filcycl. RUKK,,.)R =IK, Dern. F.*336-41.:)F:.Hp.:):REIC:(HS).SEK-,.R=KKjS (1) F. (1). *370-21-24.3F. Prop *370-31. F:KEFMcycl.RECKa-t'KK,.).RIK,,ECnv"Ka [*331-33. *370,22] *370-311. F:Hp *370-31)R I KICE6Ka 'Dem. F.*370-31. D)F: Hp. KKIR 6Ka. [*330-5. *370-26]:).R'KKE Ka:) F. Prop

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464 QUANTITY [PART VI *370-32. F:K/eFMcycl.REc.).D'R=EL'R=s'UI"xK Dem. F.

*50-5-52. F F. D'IK = 16'IK = S'P'K (1) F. *370-26. *330-52. D): Hp. D. D'KK = UfKK = s'U"IC (2) F. *370-31. *330-52.) DF: Hp. B E Ka - tf14.).D'(R i K) = S6ZL"'K. [*330-52.*34-36]). D'1 = sA'"C (3) F. (1).(2). (3). D F. Prop *370-33. F: K E FIi cyci.. D"ic = U"1I. Kc e FM connex [*37032. *334-42] *370-34. F: K eFM cycl.). UD, connex [*370-33. *336-62. (*336-011)] *370-35. F: Hp *37031.). KKUCR., (RUKKK) [*3703. Transp. *37034] *370-36. F: Kc FM cycl..cal UhE connex. 'Ka 1 UKK= K Dem. f.*336-41. F: Hp. D. C',ca1 UK C I (1) P. *370-34. D F:. Hp.R, Se Ica. R # S.): R(Ka 1 UK) S. v..(SalK UK) R (2) P. *336-41. D F: Hp. Re6Kca-S=hK.:. R (c1 UK)S (3) F.*336-41. F: Hp. SK.a T R = IK..S(1 UK)R (4) F. (2).(3).(4).)F:. Hp. B, Scc. R#.:): R(cal UK)S.V.S(~alU`K)R (5) F. (1). (5). DF. Prop *370-37. F: K EFM ycycl. Ic. UK cSer [*370-11P1P36] *37038. F:xEFMcycl. BR,SexK.).RIS=SIR [*330-561.*370-32] *370-4. F: Ke FM cyci.).Cnv'c=K,= "KKK Dem. F. *37031. *3305. F F: Hp. 1K4 "(Ica - t'KK) C Cnv"K (1) F.(1). *370-26.)F: Hp.:).KK I "K C Cnv"/ic (2) F.*37031126. D F: Hp.R /c.R.R IKKE K [*370-26].(S). S/K. R = S i K,. [*330-5.*c37~6 6 Re KK "Ke (3) F.(2). (3)) F.Prop *370-41.:. K CeFM cycI. R, SCe K.D: (K, I R)VK(KIS). - E. RUhS Dem. F. *336-54. *370,33. F:. Hp-.):(KK I R)&(KK I S). T.(T). TE/Ka. KK IR=TT KK IS. [*3305.*370,26].(aT). Tc Ica. R = T S. L*336,41].RUKS:.D F.Prop

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SECTION D] ELEMENTARY PROPERTIES OF CYCLIC FAMILiES 465 *370-42. - K FE FM cycl.). n Cnv"'K = ffIK v K, Dem. F. *370,22. F: Hp.1 RE Ka - VWK D. R I K, e K. [*.370-311.Transp]). 1,E Ka - 'KK (1) D():. Hp. H: R, ReKD. R)e' Vt'KV (2) F. (2). *370-26 D F. Prop *370-43. F K HeFMcycl.. K "'Kab= Cnv ""K- L'K, [*370-4] *370-44. F: KE FMcyclI. D. Ka n K, I ""Ka=A [*370-42-43] R. & W. III. 30

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*371. THE SERIES OF VECTORS. Summary of *371. In this number, we begin by defining the relation W,, which takes the place, for cyclic families, of the relation V, defined in *336. The definition is *371-01. W,= K I; U C Ka U a Df Then if K is a cyclic family, W, is a series (*371*12), and its field is K v Cnv'"c (*371*14), which =, since K has connexity. It will be observed that V, is not a series if K is a cyclic family; we have e.g. I,,VK,,. K,V,I,. The above relation Wc is constructed so as to make a barrier at I,, thereby preventing the relation W, from being cyclic. If P, Q are both members of Ka or both members of K, I "K, PW,,Q. -.(aT).Te ca. P=Q T (*371-15-151). Most of the properties of W, depend upon the fact that Ka 1 U, is transitive, in virtue of the definition' of cyclic families. If K is any connected family, we have al U, etrans.-: P,Q,QR,P Q | R eKa. R IC. pQ. P Q Ka (*3712). This proposition is required for most of the subsequent proofs in this number. It leads at once to *371-21.: K e FMcycl. P,Q,Q R, P IQ R Ka. Re c.).P Q Ka Most of the propositions of this number are concerned with the circumstances under which we can infer (P I R) W (Q I R) from PWQ. We have *371-31. F:. K eFMcycl. Re K: P Ka.. P IR, e Ka:): P W.Q. (P I R) W, (Q I R) Another useful proposition is

*371-27. F:. K eFMcycl. P, Q e ca. *: P W Q. -. QWP *371'01. WK=KK;UKCtKa* UKtca Df *371-1.::. K FMcycl.::PW,,Q.:.P,Q e K, I"a: (aR, S).R, Se Ka. RUS. P= K I R. Q= K S:: P, Q e a. PUQ:v: P K, | "K. Q Ka [*202-55. *370'34. (371-01)]

SECTION D) THE SERIES OF VECTORS 467 *c371-11. F: Kt e FMM. KE K.D.(K 1)r K, e I -+ Dem. F. *330-31 F:- Hp. R, SE. K R = K IS.. R =S::) F. Prop *371'12. I-FE:FM cycl.:. W, E Ser [*3703744. *37111. *20421-5] *371-13. F-:IK E FMcycl.). W =YK, (Cnvlic - t'K,) U, c [*370-41-43] *371414. F: Ice FM cycl.:. C'C4W = K V Cnv"K = Ka v KK I "Ka Dent. F-. *202-55. *370-34. *16014.:) F: Hp.:). C' W=KKvI ""Ka V Ka [*370-43] = K V Cnv"OcK: D F Prop *371415. F:. K c FM cycl. P,Qe. Ka) D: P WKQ. E. (2tT). T cKa. P = Q I T [*37044. *336-41. (*371-01)] *371-151. F-:. KEFMcycl. P,QEKK " Ka.) PW,CQ.. (aT).Teca.P=QIT Dem. F. *370-44. *336-41.) F:. Hp.:): PW,,Q.=E.(gR,S,T).R,S, TEKa. R-SIT.P=KCIR.Q=KKIS. [*370-26] =.(aT). TiFKa. P=QIT:.)F. Prop *371-152. F-:KEFMcycl1.PEKKI "Ka. QEKa.).PWkQ [*371'1] *371-16. F-: K EFMcycl. P c K. PWcQ.)..Q6EKa [*370-44. *371-1] *3714161. F: Ke FM cycl. QE KK I ""Ka. P W Q. P EJ4X ICC"Ka [*37044. *371-1] *371'17. F-: K FM cycl. Q, T E Ka. (Q I T) WKQ. (Q I T) WKT [*371-15152] 4 -*37118. 1-: K eFMcycl.) K. KK II " =11Ka. W,'J4K= Ka - tCKK [*371-15-152. *370-31122] *371-19. F-:. Ke FMcycl. P h I.: PWK,. K.K,,WP [*37118. *370-43] *371-2. F-::KcFMconx.):.Kaj U, etrans. P, Q, Q B R, P I Q I RE Ka. RE K.: P,Q,R P j Q E Ka Dem. F.*336641. F-:Hp. H):T(Ka 1 U,)S. S(Ka 1 Uc) R. (HP, Q). P, Q, S, TE Ka. REK.T=PlS.S=QIR (1) F. (1). *1321.) F:: Hp.::. Ka 1 U, e trans. P, Q,QIR,PIQiR EKa.-ReK.)p QB.(PlQiR)UcR (2) F. *330-315 1.) F:.Hp.P,Q,REK. MEKa.P)QjR=M jR.).PIQ=M (3) F.(3).. *336-41.)F-:. Hp. P, Q, R, PIQ RE K.): (PIQR) UKR. E. P I QEKa (4) F-. (2). (4).:) F-. Prop 30-2

468 QUANTITY [PART VI *371-21. F:KceFMcyc1.P,Q,QIR,P QIREIca.REK.:). PIQe~ca [*371-2. *370-1] *371-22. F:xeFMcycl.P,R,PiReia.PWKQ.:).QIRE Ka Dem. F. *3.71-15-16.) F: Hp.:). (ST). Q, TE Ka. P = Q IT(1 [*371'21] D. Q IREKa::) F.Prop *371-23. F:1ceFMcycl. TW,,S.:).TW,(S 7) Dem. F.*330-31.*370-38. DF: Hp.D. T= SI S I T)(1 F.*371-15516. DF:Hp.Tcxa.~l.SjTcica (3) F. *371-152. D F: Hp. T,,e Ka SITE Ka.). TWK,(S IT) (5) F. *371P151-161. D:H.S-, c.D,. c. SITeK (6) F.(1).*3 71-151. DF: Hp. T, S IT,-, e a.Se Ia.D. TWK,(S IT) (8) F.(4). (7). (9). DF.Pro p *371-24. F:xEFMcycl.P,R,P RExca.PWKQ.:).(PIR)WK(QIR) Dem. F. *371-15-16. D F: Hp. D. (ST). P, Q, RP R, Te/ca.P=QjT. [*371-21.*330-5]). (HT). PIR, Q IR, TexIa.P IR Q IR IT. [*371-15] D. (P IR) W.(Q IR): D. Prop *371-241. F:xcEFMcycl.P,RElca.PKIRE-sexc.PWQ.:).(PIJR)W,c(QI?) Dem. F. *371-152. D F: Hp. Q IREIca. D.) (P I R)W,, (Q IR)(1 F. *371-15.) F: Hp.QIR Ka:.(T.TIc. PIR Q IRc a R QI1R T [*371-151]:).(P IR) W,(Q IR) (2) F. (1). (2).:) F. Prop *371-25. F:,eFMCYCI.P,ReKa.PWQ.:).(PIfR)WK(QKR) [*371P24-241]

SECTION D] SECTION D] THE SERIES OF VECTORS46 469 *371-251. F: Ke FMcycl1.R, R IQE Ka. PW&Q.-). (R IP) W.(R IQ) Dem. F. *371P25. Transp. *371-12.) F::EFMCYC1.,P,ReCKa.(Q IR) W,,(PI R).:). QW,,P() M P~ Q F. Prop *371-26. F:.KE6FMCYC1:P,QEICa.V.P,QI-.JEKa::): Dem. F.*371P25.*370-26.) F:Hp.PEKa. PW,,Q.).(K,jIP)W,,(KK Q) (1) F.-*371L251. *370P26.3 F:Ilp.QEKa.-(KIP) WK (K,, Q))P WQ (2) P, Q* F:.Hp. P, Qr.-eKa.)P WKQ E(K. P) WK(K. IQ) (4) F. (3).(4).) F. Prop *371-27. F:.KeFMcycl.P,QEKa-.):PWCQ.EQW,,P Dem. F.*371-15.)F:. Hp.:):PW,,Q.=.(HT). TEKa.P PQI T. [*370-33].(r2T). TE Ka P I T. [*371P151-19. *370A43] ~.QWIP:.) F. Prop *371-3. F:KeFMcycIl.RE Ka.PIRt-.exa.PWKQ.:).(PIR) WK,(QIR) Dem. F. *371P27.) F: Hp. D. QWCP. [*371P27] D. (P IR)WK(Q IR):)F.. Prop *371-31. F:.KceFMcyc1.REica:Pe~c.v.PJRt%-IeiC~a:): PWKQ.:).(PJR)WKc (QJR) [*371V253]

*372. INTEGRAL SECTIONS OF THE SERIES OF VECTORS. Summary of *372. The subject of this number is that section of W, which consists of vectors not greater than the vth part of the whole circumference of the cycle. This is defined by means of WK, as consisting of those vectors which (taking W, as "greater than ") are such that R+1 is greater than Ra so long as u < v. It will be seen that so long as RY and all earlier powers of R do not exceed I,, R satisfies this condition; but if R E K,1 J "ca, while R'+le ea, we shall have RCWKRR+~. Thus our definition selects those vectors which, starting from any origin,.do not, by v repetitions, take us farther than once round the cycle. The definition is *372'01. v, = (K v Cnv"K) n R (o < v. 4 0.. R+1 WR') Df We then have = c v Cnv"c (*372-11), 2,= Ka (*372-13), /^ <v.)v, C/C,, i.e. v, diminishes as v increases (*372-15); v > 1.. v C "a (*372'16). An alternative formula for v,, sometimes more convenient than the one given in the definition, is (assuming v > 1) v, = eKa n P (a < v.L:O. P$L+ e Ka *).D PL e Ca) (*372-17); i.e. so long as p, < v, either P' comes in the upper semi-circle, or P,+1 comes in the lower semi-circle; that is to say, the step from PA to PA+1 does not cross I,. For an even number (not zero), this leads to a simpler formula, namely (2v),= Ka^ P (,t. / 0.),. Ps e Ka) (*372-18). We have next a set of propositions leading up to *372-27. F:. c e FM cycl. v e NC ind - 'O. P e,,. P W,,Q.: whence, since W, is a series, we obtain *37228. F:. c e FMcycl. v eNC ind - O. P, Qe,.:): Pv = Qv. P = Q It is largely owing to this proposition that v, is important. In virtue of this proposition, there is in v, at most one vector which is the vth submultiple of a given vector. We shall show later that, if K is a submultipliable

SECTION D] INTEGRAL SECTIONS OF THE SERIES OF VECTORS 471 cyclic family, there is at least one such vector; hence there is a unique vector in v. which is the vth submultiple of a given vector. This does not hold in general for larger classes than V,. A specially useful case of the above proposition is obtained by putting v= 2, which gives, in virtue of *372-13, *372-29. F:.,eKFMcycl.P,Qe ICa.):P2=Q2.=E.P=Q The remaining propositions of this number are concerned in proving that VK is an upper section of WK, i.e. *372-33. F: K e FM cycl. v e NC ind.). W'vXC v, *372-O1.- vK =(K uCnvl'x) R(o-<v. -O.)U.R,.RW+'WR") Df *3721. F:.RevK.=:.Rc vCnv",K:o-< v.4O.:). Ra+'1WKR [(*372-01)] *372-11. F.I 1, =Kv Cnv"IIK [*3721. *117-53] *372-12. F: Ke FM cycl. Re K I C"'Ia R WKR2 Dem. F. *371-152.:): Hp. R2 E Ica. R WXR"(1 F.*37O-44.:tF:p.R2 R, R 2 6 Ka. Rena~R=RIR2. [*371'151]).RWKR2 (2) F. (1). (2).) F. Prop *372-121. FI- e:FM cycl Re a.). R2 WKR [*371-17] *372-122. F:. Ke FM cycl.): Re..R- R WKR [*372'12-121.*371-12] *37213. F K: e FM cycl.). 2, = Ka [*372-122] *372-14. F: K e.FM cycl. K. KN e 3,, Dem. F. *371-152. 3 F: Hp.:). K,2W K3: F F. Prop *372-15. F p v.. vCuK [*372-1] *372-16. F:KeFMcycl.v> 1.).vca [*372-1513] *372-17. F:xeFMcycl.v>1.). vt= Ia n P (/A < v., a 0. Te+1',Ica. Pi Ic Dem. F. *372-116. *371-16. F:Hp.). VKCKanP(<v. +0.P& +I'c&.),.P# e ca) (1) F-.*371-15.)F: Hp. P, P*, P*+' eKca.). P+' WP*L (2) F. *371-152. ~F: Hp. P,Th'L eKa. PI+1eI-)'.h +a P1+' WTcP* (3) F. (3713151. F: Hp. P) Kan P(*< P*.4eIL+ Ka IA). PIe)Cv* (4) F. (2). (3). (4). F t:. Hp. P e Ka P*P e Ka. V. Pi&' E Ka~::) - PA+' W,,p* (5) F., (5).*3722~1.:)F:Hp,). Aca P ((< <..rc-O. PiL+'ea-.,,-PIAI.,)CV,((6

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472 QUANTITY [PART VI Dem. F.*372-1.*371 12.) F: Hp. Pe(2v),.).P V4TPV [*372-122].P" eiKa(1 F. *3 7115,152.:)F: Hp. P, PIZ6Ka.:).PI~I WKP- (3) F. (3). *371-25.:) F: Hp. PL,])i+i, P-0 6 a.: PPg+p+1WKPIA+p (4) F. (4). %P a',,<V.U +0,,Pa K:: /,tL+ 1 v V.P ~ V.). PI&+P+I WcPfL+P: [*372-1]:):P E (2v)K, (5) F. (2). (5). DF. Prop *372-19. F:KEFM cycl.), Ve NC ind - tO. P e(Ikv),,.).ThEI6VK [*372-1. *371-12] *372-2. F: Ie FMcyc1. v eNC ind. Pe v, qk v. o- < i +0.-O.) TWKPO' [*372-1. *371-121 *372-21. F: KeFM cycl.ve NC ind. Pe v, 2g, ~~v.,iktO. Dem. *372-22. Dem. p12,& WKPh. ThA C 1%C (1) (2) F. *372-2. D F: Hp. D. P2gW,,PI4. [*372-122]:). PIE Ka F. (1). (2). D F. Prop F: xeFMcycl. PWKQ. P, P * 6Ka.TPhW,/2y.).Th+1W*+cQ 4" F. *.371-25.)D F: Hp. D. IL)4+i W.P I f [*371-25] D. PI *W?I F. (1).(2). *371-12. DF. Prop (1) (2) *372-23. F:KeFMcycl.veNCind. PEv,,.2pt,<v.It4:O.PW,CQ.:). PIL+IWQIL+ [*372-2122. Induct] *372-24. F:. KceFMcycl.0.E6NCind-tfO.Pe(24-).PJIQ.): Dem. F.*372-21-23.:)F:Hp.4~0..fl~..),,. P'7PQ QteIca. Pt W KQ e.P'qW KQ'. [*371-25],) *J+IlWPn I Qt. P17 Qt WMQt+ni [*371-12]) Jt+'qWQt+': F.. Prop

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SECTION D] INTEGRAL SECTIONS OF THE SERIES' OF VECTORS 473 *372-25. F:. KEFMcyel.oreNCind-,1'O.,.Pe(2o-+1),,.PW,,Q.'D: p<,2o-.p+O.).P#LWICQI [*372-24-15] *372-26. F: /-F~cycl. o-eNCind.P,6(2 +1)',C. P W,CQ.*) P2071lWkQ2o*1

Dem. [*371-3])P2,7+1 E P]217+1 WKPT Q7T+I (2) F. *372-21.*371-15-151-152.) F:. Hp. D PT TlWP7 F.(2). (5). *371-12.:)'F: Hp. P2c~r~I Ka.,TlWQ2+ (6) F. *371-16.*372-1. DF:Hp. P2T+'eK/c.1)2.cK, (8) F.(6). (7). (8).)DF.Prop *372-27. F:.KEFMcycl.veNCind-tfO.P'CVK.PWCQ.:): pv. p+0.).PThWcQA [*372-2425-26] *372-28. F:.KeFMcycl.vENCind-t'O.P,Qev,.):P,,=Q,,.-.P=Q Dem. F.*371-12.) F:. Hp.PtQ.):PW,,Q.v.QK [*372-27]:) PI WKQv. v. QvI4&P": [*371-12]): P-,+ Qv 1 F.(1).Transp.)F.Prop *372-29. F:.,cEFMcycl.P,Qeica.):P2=Q2.=.P=Q [*372-2813] *372-3. F:xKeFM cyc1a e NC ind -t'O. Pe(2o-,. P WKQ.)Q c(2ci), Dem. F. *37 2-1827. D F:.Hp.)3:/..),s.P* elKa.ThuW~cQ". [*371-16] D,4 Qt E ca [*372-18]): Q e va:.) F. Prop *372-31. F:.KeFMcyc1.oeNCind-ffO.Pe~a.:):PWKP~o.) P2,7+1 /Ka Dem. F. *301-23.)F: Hp.D. P=ThfP20 ~'+ (2) F.(1). (2). *37115. D F: Hp. P WKPT. P+',E Kce::) F. Prop

474 QUANTITY [PART VI *372-32. I~:KeFM cycl.a~eNC ind.Pe(2cr+1),,.PWcQ.). Qe(2or+1),c Dem. F..*371-16. *372-27-1.3 F: Hp Q2'r".'eKx8.)P~r"-~' e Ka. [*372-3l.Transp].P"WKP. [*371-27] D. Q2WWKP. [*372-3l.Transp]) + lea (2) [*372-17] D: Qe(2ar+ 1),K:. I-F. Prop *372-33. F:Ke FMcycl. 'e NCind.). WM,'vX C Pg [*372-332]

*373. SUBMULTIPLES OF IDENTITY. Summary of *373. The purpose of this number is to prove that, in a cyclic submultipliable family, there exists a unique vector which is a member of v, and satisfies RV=I. This we call the "principal" vth submultiple of I,. It is the smallest vector (other than I,) which satisfies Rv =I,. The proof of its existence proceeds by several stages; the problem is analogous to that of the construction of a regular polygon. Suppose the cycle divided into v equal parts. Then a vector which takes us from any one point of division to any other is a vth submultiple of identity. If v is prime, every such vector will have every power less than the vth different from,,; but if v has factors, say p and a, if R' = I,, (R^)r = I; thus RP, which is one of the vth submultiples of identity, has a power less than the vth which is equal to I,. We define (I,, v) as the class of those vth submultiples of I, which have no power less than the vth equal to I,; more generally, we put *37303. (S,y)=P(Pv=S: a <v.4O.),0.P'+:S) Dft We then have first to prove the existence of Ka n (I,, v) when K is cyclic and submultipliable. For this purpose, we put *373-01. M,, = QP (QKa. Q = P) Dft I.e. Mv, is the relation of a vth submultiple of P to P, when the submultiple of P is a member of ca. It is to be observed that although K is submultipliable, we do not know to begin with that I, has submultiples which are members of Ka, except in the case of K,, which is half of 1,. Owing to this, we proceed first by bisection, i.e. by means of the relation M,,. We prove that the process of bisection can be applied endlessly to any member of Ka, and always gives new terms (*373*14'13), hence it gives a progression starting from any member of Ka

University of Michigan Historical Math Collection: Principia mathematica, by Alfred North Whitehead ... and Bertrand Russell.

(*373'141), and therefore the existence of a cyclic submultipliable family implies the axiom of infinity (*373'142); also we prove that v bisections starting from a member of Ka give a member of (2v+1), (*37:315). Hence, taking K, as the member of Ka to be bisected, we arrive at I = 2t+' en. t i! s a nt (nm, e) (*373-17). In order to extend this result to numbers not of the form 2v+1, we have

 first to prove that there are p/th submultiples of identity. This we prove first for numbers of the form 2v + 1, then for (2o- + 1) 2v + 1, and then for 27 (*373'21'22'23); hence it holds generally, i.e. we have *373'25. F: Ke FMcyclsubm. e NCind - '0 - t'1.. (Q). Q Ka. QL=I Next, we prove that, if Re cK and R*=R = TK, then u/, v have some common factor p such that Re (I,, p), i.e. such that RP is the earliest power of R which is IK (*373-3). Hence if / is prime, and RT=I,, it follows that no earlier power of R is I,, i.e. R e(I1,,u) (*373'32), and that, if R (I1,p) and R= KI, then,/ is a multiple of p (*373-33). We now make a fresh start with the general relation MK. Owing to *373*25, we know that I, e (CMfK. Also since K is submultipliable, IaC (I'MV. Hence if a is any inductive cardinal, IK 6eMVKa (*373'404). Also it is easy to show that if v is a prime, and QMVK^IK, Q&V is the first power of Q which is I,. Hence when v is prime, Ica n (I,, Pa) exists (*373-43). In order to extend this result to numbers which are not powers of primes, we prove *373'45. F: K e FMcycl. pPrm. R e(I,p). Se (IK,).. RS (IK,pa) Hence by the help of a little elementary arithmetic we arrive at *373'46.: ic e FMcycl subm. p e NC ind - ' - '1. D. g! ca n (IK, p) Having now proved that there are vth submultiples of I, which have no power short of the vth equal to IK, we have still to show that there is one among them which is a member of v,. For this purpose, we take any one of them and consider its powers. It is obvious that it has only v different powers (*373 5), since after reaching I, the previous values repeat themselves. It is this fact which makes it easier to deal with submultiples of IK than with submultiples of other vectors. Now let R be any vth submultiple of identity, and assume that S, T are powers of R, but T is not a power of S, and TWKS. Then S T is a power of R but not of S, and TW,(SI T) (*373'53). Hence T is not the maximum, in the series WK, of the class Pot'R - Pot'S. Hence by transposition, if T is the maximum of Pot'R - Pot'S, we must have SWKT. Now since Pot'R is a finite class, Pot'R - Pot'S must have a maximum if it exists; but since S has the relation W, to this maximum, S is not the maximum of Pot'R. Hence by transposition, if S is the maximum of Pot'R, Pot'R-Pot'S is null, and therefore Pot'R = Pot'S (*373-54). Hence it follows easily that, if R e a n (,,),)the maximum of the powers of R is a member of Kca n (I, v) (*373'55), and further that it is a member of vK (*373'56). Since we have already proved (*373'46) the existence of Ica n (I,, v), we thus have *3736.: Ec FMcycl subm. v NC ind -'..!, n S(V=)A *373'6. F: K e FM cycl subr. v e NC ind - t'0. D. [! r, n S (Sy = 1K)

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SECTION D] SECTION D] SUBMULTIPLES OF IDENTITY47 477 The uniqueness of v,,nS S = 1K,) follows from *372-28, and thus the principal vth submultiple of I,, exists. Hence also it immediately follows that the other vth submultiples of h~ are powers of the principal vth submultiple, and that the total number of vth submultiples is (*373-63-64). *373-01. M1,C = QP (QE Ka. Q P) Dft [*373-5] *373-02. Prirne=NCindn'(/z=o-x,,r.:),T,:o-=I.v.a,=,a) Df *373-03. (S,vP)=P(Po=S: o0< v.a +0.),.P%[+S) Dft [*373-5] *37341. I:M2KP. Qe Fa. Q2 P [(*373-01)] *373-11. 1K Fcc.).M I-+1[*372-29] *373412. 1-: Ke FM cycl. M. C_ W, [*372-121] *373-13. F:K e FM cycl. (M2,,)o C W,- - (M21~)PO C J [*373-12. *371-12] *373-14. F:Ke FMcycl subm. PeKa -v eNC ind - vO.)E! M2,v(P Dem. F.(1). Induct.) F-.Prop *3734141. F K e FM cycl subm. P E Ka.) M21C ~ (M21,)*'P e Prog [*373-11-1314] *3734142. F g! FMcycl subtn.). nfin ax [*373-141] *373-15. F: K C FM cyci subm. P E Ka. v 6 NC ind.).M2KV'Pe (2+' Dem. F.(2).*373-1.:) F:.Hp(2).:):2o-<2P.:).R R2aT Rc+2, R2,R eKa [*371'2]). Cl Ka (3) [*37 218]) R e (29v+I)K (4) F. *372-13 D HFlp v 0.) M2K"CP e2, (5) F. (4). (5). Induct.)F. Prop *373416. F: K e FM cycl subm. v e NC ind. Q = M2,,,"'K,.) 2+=I1K: p < 21~'. p #0. D,. QP+ h Dern. F. *373-1 D F: Hp.:).Q2=KK. F.*373-15. *372-2.(1). DF:. Hp)p < 2P'+'.p4.). QP WJK (2) F. (1).(2).:)F. Prop

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478 QUANTITY [PART VI *373-17. F:e~ylui~e~n~p2+.)HKMI,,U [*373-1614. (*373-03)] *373-18. F: QeCnV"Ka.Q"=h,.:).QlExa.Qv=h, [*50-551] *373-19. F: (HQ). Q eca vCnvl"ia. Q"= ~ (HQ).Q e Ia.Qv [*373-18] *373-2. F: E EFM cycl subm. vE NC ind. P=M2~vK,"K. SE,%K. S2"+I=P. S2v = Q.:). Q2'+I Q+ QtI Dem. F.*373-1.)F: Hp.).P21"~1= KK IP. [*370o22] p2%1* + P. [Hp])D2+ 2+ [*30-37]).PtS (2) F. *301E523.) F: Hp.. Q = (S2v+I)2 S2 [Hp] =P21 S2 [(2).*372-29] + h1 (3) F. (1). (3).) F. Prop *373-21. F:KeFMcyclsubm.vENCind.,u=2"+1.:). (HjQ). Q E Ica. (?= IKc [*373-219] *373-22. F:KceFMcyclsubrn..v,creNCind./L= (2oa+1)2v+I1.:). (HQ). Q eKa. QIL=IK [The proof proceeds as in *373-221] *373-23. F:xeFMcyclsubrn.o-ENCind.,u=2o-.:).(HtQ).QESKa.Q"=Ih Dem. F.*370-26.:)F:Hp.).KKaEc. K* = h,:)F.Prop *373-231. F:.TeNCind.:):(3[o-): a'eNCind:r= 2o-. v.Tr=2o-+1 [Induct] *373-24. F:peNCind.p+f.:). (av, -). v, a- e NC ind. 2p +1I = (2a- + 1) 2v, + 1 Dem. F. *1 17'661.) F. *116-301. DF: Hp (1). D. p= p20. [*10-24] D.O'ex (2) F. (1). (2). *261-26. *263A47.) F+:. Hp (1).): (av): veX: pS>v)qe (3) F.*116-52-321.) F: P=T21'. 7= 2-.). P-=a2I" (4)

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SECTION D] SUBMULTIPLES OF IDENTITY 479 F-. (3). (4).:)F:.Hp.): (v, Ti:vx, r eNCind. p = Tr21: p >. (HT-). p =-24 e-' (u-). T= 2o-: [*116-52'321] D(av, a-). vi, o-e NC nd. 2p + 1 = (2o- + 1) 2av+1 + 1:.Prop *373-25. F:/KeFMcyclsubln. peNCind-t'0-tf1.). (HiQ). Q e Q& = 4t [*373-22-24-23-14] *373-3. F:Ke FM cycl.p 4:0. v 0. R e Ka R'% =R~I',.). F.*300.23.:)F:.Hp.:):(ap).pt0.RP=IK,:o.<p.cro.#o) U.RT~II (1) F.*301-2. 3F:Hp.R7P=h,.D.ptI (2) F.*302-25. F:HP.p eN`C irid -t0.:).

F.*3O1P23-504.) (,,,)~X+9v7~q<.< 3 F:H 3.R 1,-/t=a 8,p+(4) F. (4).) F:. Hp(4): a- <p. a#0.)D. Rlr4lh: J4=aP+f3.v=ryp+8:)./8=0.8=0 (5) F. (3). (5).:)F:. Hp: p4O.R,:0-< p. otO.),. Re'+h:). F. (1).(2). (6). (*373-03). F..Prop (Hfa,y)., = ap. v= yp (6) *373-31. F:xceFMcycl.Reica.,u#0.xi#.R*=Rv=I.:).4...,(,gPrmv) [*373-3] *373-32. F: e.FM cycl.R eKa.,a e Prime.1?R'=hD.1? e (I1,M) [*373-31. Transp. (*373-03)] We assume here that a prime number is prime to all numbers less than itself except 1. This follows at once from the definition. *373-33. F:K EFMcycl.Reica r(Ih,p).RP'h=.:).(3r).Lt=pT [*373-3] *373-4. F:QMI,ICP..Q e/ P =Q [(*373-01)] *373-401. F: E FM cyclssubm.vie NCind-tff0.). I,, e (I'M,, [*373-25] *373-402. F:K eFM subm.vie NCind-tff0.).KaCC(I'M,1, [*373-4] *373-403. F: xe N`Cind -tf0.).D'1MVK Cica [*373-4] *373-404. F:KeFMcyclsubm.vilaeNCind-tf0.). I,,eq'M,,, [*373-401P402-403. Induct] *373-405. F:v, a eNCind-tff0. QM,,IK.).Qv = I, [*373-4. Induct]

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 *373-406. H. ICEN d 0 ~.M" [*3734. Induct] *373-407. 1- v,a, y eNCind -tf0. RM,,a+V1K,.). R~"M,'MY [*373-406]1 *373-41.' ~: ~,8ECidt,.M."'.MI.<8:.+ Dem. 1-. *373-405A407-403.)D F: Hp. D. Q~= I,,. RI / cK: D F. Prop *373-42. F:iceFMcycl.vePrime-ffl.cENCind. QMvoaI,,.o- < Vol. O-40) QC, =jK Dem. F. *373-405. *300-23.)D F. *373-33-405.)D [Hp])(a/3) P z (2) F.*373-407. DF:Hp.i3< a. D. Qs,'#h (3) F.(2).(3).:)F:Hp(2).).p=vdx (4) F. (1). (4).)DF. Prop In obtaining (2) of the above proof, we assume that if Vis a prime, and p'r is a power of V, then p is a power of v. This is easily proved. *373-43. F: KcEFMcyclsubm.vePrirne-t'1l.aeNCind-ffl.:). 2[! /ca A (I," Va) [*373-404-405-42] *373-44. F:,yPrrnp.fyPrmc.o.).7yPrmpo, Dem. F. *302-1:)F:.cyPrmp.,,(fyPrmpa-). areNCind.:). (HT,a,j1?).reNCi'nd-t'0-ffl.ly=a'r.po-=,8T (1) F.*303-39.)F:Hp (l). TeNC ind - iO- t`1.r y aT -po. =/3T.),yp= aojf38 (2) F. (2). *303-341. DF: Hp (2). aoK Prm/3. D.7y= ao (3) F.(3.).*302K1.)F:Hp(3).a+#1.:).,yPrmcor) (4) F. *113-621.:)F:pe NC. a= I y Prm.pa).).(ryPrm p) (5).F. *302-36. D F:Hp (2). (ao-Prm p3).). F.- *303-39 V F: Hp (7). f4Prm q =~ ao. 8= aojr3 = [(2).*303-341],= P=q [Hp]).apo- =/3ry =ap ~T. [*126'41] O).= T (8)

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 - [*3021.Hp]) '(,y Prm, o-) (9) I-. (6). (9).) Hp (2)). QyPrrmo-) (10) F.(11).Transp.)FProp *373-441.:pra(8)p=8:().3 & Dem. F-.*126A41.) F: p3p3 (a p=44(=] ~Pm~.) I =q no-. Prmqc (.1 rma F-1. M:H 1. 13.~.Ia x,8 [*302-1]:). (~ Prmqo-) (2) F-. (2). Transp. (1).) F: Hp(1)). =1 (3) F. (1). (3).) F.Prop *373-45. F:KEFMcycI.pPrmoa-.RE(IK, p).SE(hK,o-).).RjSe(hK PO-) Dem. F. *370-3-3.) F Hp.).(J IfS)PO,-h (1) F. (1). *373-31.: Hp. (R IS)V= IK. fy + O.: (ry Prm. Pa-): [*373A44]): (,Jy Prm p) v. -J (Prm o-) (2) F.*370-33. *301-504.)D F: Hp (2). p= aT. y = fir.. h=(R I S)aPr SEZpr =p [*370-33]) (H8. P8=8 [*373-33]).(Hpi) - fir = paT. F. (3). (4).) F: Hp (3).).(3v). y vpc-. vto (5) Similarly F:Hp.,(fy Prmn o-).) (rv).y =vpo-.v 0 (6) F. (1).

(7). *117-62.)3F. Prop *373-451. F:.peNCind-t'0:c,(Uv,a).vePrime.p=v":). (iv). p~ Prma v. p < p. vi < p. p= Dem. F.- *261-26. *263-47.) F: Hp. -) - (ay, a). fy e Prime. p e Dlcxy 7. p~ ED'x,, r+ p + a [*37,3-44.1nduct]:). (Hpy, a,8) fi- y e Prime. p = yai. fi Prm fya. (341I:) F - Prop R.& W. 111. 31

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482.. QUANTITY [PART VI *373-452. F:.vEPrime.a6NCind.)v,,a,.k(va):, aPrmv.4~.4w.),s,,v. Ob(,Lv):):peNCind-tfO.)P.Ob(p) [*.373-451] *373-46. f-:KE FMeyclsubm.peNCind-ftf0-t1.:).2!K~a(IK,,p) {*373-4-345-18-452J *373-5. H: KEFM cyc1.vE NCind.Re Iari (I, v).Pot'R osv Dem. I -.*302-25. *301-504.)D [*120-57]). Nc'Pot'1?,, v(1 F. *301-23.)F: Hp. p< v.a-< p.:). RlrjRp= RBP, [*330-32]) RP + Rlr (2) F - (1). (4).)F. Prop *373-51. F:KEFMcycl.1?exan(h,,,iv).:).RThE(JI, z).Pot'BR'ev Dem. -. *301 504.) F-:. Hp.:): (RAt)v= I,,: a-< v. o-+0.). (R/s) 17#I,,:.) F-.Prop *373-52. F:tcEFMcycl.ReK~a(K, v). gPrm v.:). RIL e (1K, v). Pot'Rh - Pot'R Dem. F. *373-33.)I:Hp.R*%e(I, p).)(r,_T). Ip = PTi. [Hp]:). p v (2) F'. (3). *373-51.) F: Hp. D. Nc,'Pot6RA=- Nc'Pot'1? = v- (4) F-.}4). (5). *120-426. Transp.) F: Hp.). Pot"R/A= Pot6-R (6) F. (3).(6).)-. Prop *373-521. F-:/ ceFMcycl.Re(Kca vCnv"icKa). veNCind. - R=h,.:).RePot"R Dem. % F I. *301-2. *13-14;. D)FHp.)D.v+0 (1) F,.(1)..*301-21. D)F:Hp.)D. R= Rvc-:)DF. Prop

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BECTION D] SECTION D] SUBMULTIPLES OF IDENTITY48 483 *373-522. F: Hp *373-521.5S, Te Pot"'R) S T ePot']R Dem. F.*373-521. F: Hp.).S EPot'S [*91-6]). Se Pot']R. [*91-343] D. SJITePot'?:)DIF. Prop *373653. -: Hp *373-521.8, TePot']?. T ePot'S. TWKS.D. T WK,(Sj T).S I T EPot'R -Pot'S Dern. F*373-522. D) F: Hp.). SI TePot'] (2) F. *91-36. Transp.)DF: Hp.)D. S ITc,EPot'S (3) F. (1). (2). (3).)D F. Prop *373-531. F:Hp *373-53.). ' T =max (WK,)'(Pot']?- Pot'S)} [*373-53] *373-532. F: Hp *373-521. SEPot']?.T7= max (IY)'(Pot'R?- Pot'IS).:). SWKT [*373&5.31. Transp. *371412.] *373-533. F: Hp *373-521. Se Pot']? E!max (WK,)'(Pot'R - Pot'S).)D I S = max (WK,)'Pot']?} [*373-532] *373-54. F:Hp *373-521.S =max (WK,)Pot']?.).Pot']?= Pot',S Dern. F.*373-533.Tra-nsp.)F:Hlp.). E! nmax (WK)'(Pot']?- Pot'S) (1) F.(1).*3733-5.*261P26.Transp. F:Hlp.).Pot,']?- Pot'S= A (2) F.(2).*91-6.)F. Prop *373665. F:KEFMcycl.veNCind-t'O.]?EiarM(JK,, V). SZ=max(WK,)'Pot']?.D.SE6(IK,vP) Dem. F.*373-3-5.)F:Hp.:).(3p).peNCind-t'O. SE6(IK,p).Pot'SEp (1) F.- *373-54-5.) F:.Hp. D): Pot'SE v ': [*100-34] D):pe NC. Pot'S cp.)p =v (2) F. (1). (2).:) F. Prop *373'56. F Hp *373-55.) S E VK Dem. F. *205-21 D)F:Hp Qe Pot']?- t'S)QWKS (1) F. (1). *301-21.)F:Hp.ac ENC ind. Sa~I#+S.:Sa~I WKS. Sa~I =Sa S: [*371'15])Sa+1E Ka.).SaExKa (2) F. (2). *373-55. D)F:Hp.:a. a <V. Sa+I EKa.)D. Sae Ka (31) F. *371-16)F:Hp.)D-S E Ka (4) F. *301P2.*13-14.)D F: Hp.)D. v >1 (5) F. (3). (4). (5). *372-17.)D~. Prop

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484 QUANTITY.PART VI I *373-6. H:Ke.FMcyclsubm. veNC ind- t/O.:). a! VK MS (S"=IK) [*37346-56'5 *261-26 *372K11] *373-61. I-: Hp *373-6 v- n (SI'=h1,) el1 [*372-28.- *373-6] *373-62. E-:Hp*373-6.S~vK,.S"=IK,.D S e (1,, v).- PotS Cs P (PLO = F1) (Kcv Cnv"ic) Dem. F.*373-55'56-61.)I: Hp -D.) se(1, v) (1) F. *373-5654. FI:Hp. Kvax. T= max (WK)'Pot'R. D. S, Tc v, v SI. "Re Pot'T. [*372-28] D. S= T. RePot'T. [*13-12] D. R e Pot'S (2) F-.*37 219. F.H.) LR Sr6ZK F-. (1). (2). (5).)ID F.Prop *373-63.F I: Kc FM cycl subm.ve NC ind - t'O.]P (Py = 'K,) Ai (K V CnV"xK) = Pot'(?S) (S E VK, 1 " K) [*373-6162] *373-64. I:xe.FMcyclsubrni. veNCind-t'O.:). Nc'{P (PI' = I,,) A (Kc v Cnv,"i/c)} = v [*373-635]

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*374. PRINCIPAL SUBMULTIPLES. Summary of *374. In this number we prove for any vector what was proved for I, in *373, namely that, if v is any inductive cardinal not zero, and R is any vector, there is just one member of v, whose vth power is R. This one we call the " principal " vth submultiple of R. The proof of its existence is as follows. Assume R is a non-zero vector, and Q is a vth submultiple of R. (Q exists provided we assume that K is submultipliable.) Let T be the principal vth submultiple of IK, whose existence has been proved at the end of *373. We wish to prove that there is a vth submultiple of R which is a member of v,. By *372 33, Q is a member of v, if TWQ. But if QWT, then T must have a last power TV such that QW KT, and for this value of a we shall therefore have T+1 WQ. (We cannot have Ta+ = Q, because if Q were a power of T, we should have Q =I, whereas by hypothesis Qv =R.) Now if Ta+1WKQ. QW, TV, the vector TI Q must be less than T, i.e. we shall have TWK (TI I Q), and therefore Tr I Q will be a member of P, by *372-33. Moreover since TV = I,, we have (T i Q)v = Q" = R by hypothesis. Hence T I Q is a vth submultiple of R and a member of v,. In virtue of *372-28, it is the only vth submultiple of R which is a member of v,. Thus the existence of the principal vth submultiple of any vector is proved, assuming the family concerned to be cyclic and submultipliable. We prove also in this number that v, consists of all non-zero vectors not greater than the principal vth submultiple of IK, which is therefore the greatest member of v,; that is, we have *374'21.: c e FM cycl subm... = (WK,)*(R) (R e J. RV = IK) *374'1. F:. KeFMcycl. R, Qe a. Q" =R. Tev. T = I,.: TWKQ..Q e v, [*372-33] The above hypothesis is not all necessary for the conclusion, but is adopted because it gives the construction with which we shall be concerned. 31-3

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486 486 ~~~~~QUANTITY [ATV [PART VI *374-11. F:Hp*374-1.QWKT.:).(5ja). Tah~ 1WKQ.Q WI la' Dem. F.- *301P504-3.) F: Hp. a-c NC nd.).3 Q =+ T~ F. *373-62-5. DF:Hp.D.Pot'Tcv. (1) I [*261-26] D F. (1). (2). *372-1.)F~.Prop m min(WK)'(Pot'T n W/,Q) *374-12. Demn. F: HP *374-11. Th+' WQ. Q W,[I~. P= T1~1 Q.:). P EPv F. *371P23-16.) F:. Hp.): P ca. 1~, cIca: [*371P25] D:PW ').

PITW o+ [Ti-ansp.Hp] D: TWcP: [*372-33] D PPe c:-:.)F.-Prop *374-13. F:KC FMCyCl subm-Rc Ka.)(a[P). Pcv,c. pi,=R Dem. F.-*374-1.:)F:Hp*374l.TW~cQ.:). Qev~c.Qv=R F. *374-12.:) F: Hp *37412.:). PcevCPi,= R F.*373-6. D F:Hp. D. (aT). Tc vTv =Ic F. (3). (4).)F.Prop (2) (1) (2) (3) (4) *374-14. F:KcFMcyclsubm. RcKVCnaV"x.:).(gP).PcEVK.Pv=R Dem. F.*374-13.*373-6.) F: Hp. SC cKa.1 R= S.:). (HT, Q). T, Q EPIC. T =hI. Q = S.R = S [*372-27])(HjTQ). T, QCvK.TWCQ. (Q IT) = S= R. [*371-16.*372-33]).(:T, Q). T, QC vCQ I T cvK (Q IT) = R (1 F. (1). *374-13.-*373-6.:) F. Prop *374-2. F: KecFMcycl subm. RcKv Cnv""c.:). vI n P (Pv =R) e1 [*374-14. *372-28] *374-21. F: K E FM cyci subm.)K =(WK~I)*'(iR) (R c vK. Rv = IK) Dem. F.*374-2.)F:Hp.).E!(IR)(R6 PK Rv=IK)() F. *372-33.)F: Hp. RvC K. Rv=hK.(WC)*'cR C K (2) F. *372-152.)F: Hp.RCvPK.R" =hI.-P cvPK) Rv(W,)* P". [*372-27].R (Wic)* P (3) F. (1).(2). (3).)F. Prop

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*375. PRINCIPAL RATIOS Summary of *3 75. In this number we define a relation (FL/V)K, which is contained in (,ul/v) ~ Ic,*, but has the advantage of being one-one, and of excluding (p/o-)K unless 1u/v = p/a-. The relation(,/) is defined as holdiing between R and S when the principal 4tth submultiple of 1? is identical with the principal vth submultiple of 5, i.e. we put *375-O1. (Rtv SIk(3T).T I n rv,. R= Th.S= TI Df (Here UKrVO = I-L if~ p z.', and= VK if v >. p, by *3 72 1 5.) The properties of (/) result from *374-2. We find that, except when,uvOor =q, PIV =Vn =/i- (pt/v),c = (~/fl)i (*375-27). If Cf v '(,s/V). K V Cnv",CC/ (*375-141), 4 -and D'C(PI&V)i = (W8C)* '(pj/V)8'IK (*375-22). The principal vth submultiple of S is (1/v),'S, and its (,a/v)8'S. Also we have (1/PX'C(1l/vX'S = (/pv) K'CS (*375-15), N e vi. (1/p)~'rN'e (pv),C (*375-16), = (/4/1) (I /v),C (*375-2). The propositions (pj/v),c I (p/a-),c = WpV Xs P/a-),c and {(pjiv)/'RJ I{(p/a-)8c'R} = (,jv +,, p/a-),c'(R do not hold without limitation. The former requires either V v V * a- >. P or that the converse domain should be limited to i.e. to D'(a-/p)8,. The latter requires either pth power is ~U/v +8 p/a- <r 1/1, or R e PU(jt/v +, p/a-)K,. * Except in the trivial case when /A=O0. = 0. In this case, (/A4j') ~ K = Ai but (=~) =1, j 4.

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488 QUANTITY [PART VI *375-01. (,l),RIg)TpvR T.=" Df *375-1. F: R (l/v)WS. E. (3T).T cp, n vK.-R = ThL S=-Tv [(*375-01)] *375-11. F:KcEFMcycllpi,vENCind-t'0.:).(/4lv)KEl-*1 Dem. -. *372'28.:) F: Hp. Re KvCnv"K.c -T, WE c n R = TI= W9.3. T= W (1) F. (1). *375-1.) F: Hp.1?R v)S. R Sv kS'.).S =S' (2) Similarly F: Hp..R (pl),fS. R' (pjv)KS. R R= R' (3) F. (2). (3).)DF. Prop *375412. F IC E FMcl C. I (= V = 0). (,U/V)K, C (p/1V),~ [*3703.3] *375-13. F. = Cnv'(,at/v)K, [*375'1] *375414. F:.t, v.*K EF~cycl subrm.).D'(~L/v) =Kv~. Cnv'%c [*374-2. *372-15"] *375-141. F: z, v *K EFMcycl subm.). (pjOV)K=K V Cnv"'K [*375-13,14] *375-15. F:KEFMcyclsubm.SEKuCnv"/cK.p,veNCind-t'0.:). Dem. F. *375-14.)F: lip.:). E! (1/p)K'(1/V),K'S E! (l/Ipv)K"S(1 F (1). *375-1.)D F:. Hp.) D:M = (1/p)'(l/V),1'S. =.

(2jN).N,6vK.MEp,c.Nv'=S.MP=N (2) F. (1).*3751l.)F:. Hp.):M =(l/p4),'S.. ME (pv),.M = S. [*372-19] D. MepK. MP E V,, (MPY =S. [(2)] D. M = (I /pX',(Il/v)K' S (3) F. (l).(3).:) F. Prop *3754151. F:KE6FMCyC. NEPJ,.). N= (1/vXNv [*375-1] *375416. F:K eFMCycl subm.NE v. p ENC ind - '.). (1/p)K'N e (p4), Dem. F. *375-15-151. D F: Hp.:). (l/p)/IN= (l/pv)K'N". [*375'1]:). (1/lp),'N,6(pv):) F. Prop *375-2. F KE EFM CYCl. v, E NC ind - t'0.) (HkVX = (Pt11X (1/4) Dem. (aT). TEcIktK n v,.]R?= ThL. S = Th,:.) F. Prop

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SECTION D] PRINCIPAL RATIOS 489 *375-21. F: c eFM cycisubmn. ~j!(*t4A (p/4.),. = p/aDem. F. *3751. F: Hp. P (HlvX Q. P (p14) Q.:) (aS, T).-S e p. Kv,C Te p,jir a-, P=SL= TP. Q =S~-y =T'(1) Te p,, a-,,.P =&SL= TP. Q =S,=& Th S-=R %.R e(poa-),c A(va-)~. [*301-504]). (HR, S, T). SEFLK, eA V,. T EPI A a-K. 1? e (ka-),, en (vao-),,. P =S9t = T/P = R"". Q = 5"v =Tw=RO [*372-28]).(3B,S, T).8epacAVK Te pK A a-,K* ReQa) eA (v4-). P = S*= TP = RfL7. T7=11". [*301-504] D. (HR). 1? (Ua-)K A.P)I R"P = Rfr(2) Similarly F Hp (1).vp pa-.)D.4tua= vp (4) F.(3). (4).)D F:Hp.). wo- = vp: F F. Prop *375-22. FK E FM cycilsubm. ~tp.). D'(~t/v), =( )*,uv,', Dem. F *375-1.) F:. Hp.):B eD'(1.t/v),. (jS, T). Tep, t A, R~. = Tf S=17". [*372-15.*21-2].(HjT). Te vK. R =Tt [*374-21] (3S, T).SEvc. Sv=I s (WC)*T. R=T& [*372-27] T2 S.S 1,S,)*T [Hp] E(as). SEVK.S "-I S*t(WK)* R [*375-1-111] J (p-/,v).c'I4 (WK)* B:.) F.Prop *375-221. F:K6eFMcycl subm.. v.>) UQvK K*'v CKI *375-22 *375-131 *375-23.' F eF yl um k eNCid (av=0 1, [*375-14-141] *375-24. F: K EFMcycl subm.-(p/v)K = (p/o-)w-) ujv = p/a- [*375-21-23] The cases when we do not have pt, v, p, a-e NC ind - tf0 require separate treatment in obtaining *375-24, but they offer no difficulty.

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AGO Imij QUANTITY [PART VI *375-25.:I-Ke FM cyci subm p Prmoa-. lv = p/ a-.). (,v), =(l), Dem. F.*303 39.*302-35.)DF:Hp)(HT)-).,t= PT. v=u (1) F.*:372-19.)F:Hp.,k=pr.v=o-r.Tep,,rv,,.R=Th.S=Tv.P=TT.:). P6p,~Ao-.R=PP S-Pa' (2) I. *375-15-. F-: Hp (1). p,=p'r. v=o-,r. Pep,. =-. S= Pr'.1= (1/r),,'P.:). Tep~nv~c. =TIL S-Tv(4) F. (1). (4). *37 51.) I: Hp.) (p/uX) C (plv/v (5) F, (3). (5)).DF.Prop *375-26. I:e~ylur ~=)r~4=lOq/=:n) Dem. F-. *303-39. *302-34.)D F: Hp.,~a, v, 4:, q e NC ind.).(gjp, o-).- (p, o-) Prm (,,v).(p, a-) Prm (:) [*375-25.*303-211]).(Hp, O-). (p/0-) = (a/0Lv). (p/O-) = (4:/n)[*13171]). @t/i4 = (P/a-X (1 IF *3751-. *303-11-14182.)D F-. (1).(2).) F. Prop *375-27. F:.xEFi~cyclsubm., (Ft=v=0).,(4:=n=0).:): 4:/. (L/Pc (4/fl [*375-24-26] *375-3. F: c eFMcycl submqp, v,p, aeNC ind - tfO.) (P/0 I (la),C (Ptp/va),c Dem. F. *375-1.F: Hp. P (HavX, Q. Q (p/4-, R.:). (gS,T).Sel,k~ev,.P=SIL.Q=&".TEp~en-.Q-=TP.R=T' (1) F. *375-141-15.)D F: Hp. SE A" v, V. P = S. Q= Sv. Te p, n o-,. Q =TP. R = T%). (H{M).M =(1/p)K' S. P= rP. Q = M'P =TP.R Th Me (,p),,. [*372-28]).(HM). Me (,. p),. P =M*P.T= M".R =To (2)' Similarly FHp (1). Po- >pp.)P (pp/va),,R (15) F. (4). (5).) F. Prop

SECTION D] PRINCIPAL RATIOS 491 *375'31. F:. e FMcyclsubm.,, v,p, e NC ind - t'O: /.v.. p:). (Dm p/vlO) = (0/A),c (p/f), Dem. If P (pp/va)/ R, we have (3M). M e (pp) nr (va-),K P = MP. R = M'. The result follows by putting Q = Myp. Without the hypothesis p v. v. a- p, we have (ppl/vO-)KR = (pv/),'(p/a),,R, if R is sufficiently small to ensure (1/va),/R e (Vp),, i.e. if (al/p) 'J (W,)* R, i.e. if R e ('(p/ a),. *375-32. F: K e FM cycl subm. pt/v + p/o- <r 1/1. R e K Cnv"K.. [{(P)),K'R} I {(p/a),'R} = [{(,u/ +8 p/la)'R} The proof follows immediately from the definitions. The same result follows without the hypothesis pL/ +,p/a <r 1/1 provided R is sufficiently small to ensure (1/va),'R e (pp + va),, i.e. R e (I'(,u/ +8 pla),. VNIV. OF MICHIGAN, APR 28 1918 CAMBRIDGE: PRINTED BY JOHN CLAY, M.A. AT THE UNIVERSITY PRESS.

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